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Worksheets for Mathematics I

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Introduction

The study material is designed for students of VSB - Technical University of Ostrava.

The worksheets consist of several theoretical sheets, some solved problems and some sheets with unsolved problems for practicing. The materials should support classwork and they are not recommended for self-study or as a replacement for textbooks.

Thanks

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Worksheets for Mathematics I

Systems of linear Equations

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5 – System of linear equations

Three friends went to a sushi bar and each one bought himself a dish. The first one paid 24 coins for 1 hosomaki, 5 futomaki and 2 uramaki. The second one paid 23 coins for 3 hosomaki, 4 futomaki and 2 uramaki and the third one paid 14 coins for 4 hosomaki, 2 futomaki and 1 uramaki.
What is the price of each type of sushi roll?

The problem can be formulated as three equations in three unknowns:

$$\begin{aligned} h + 5f + 2u &= 24 \\ 3h + 4f + 2u &= 23 \\ 4h + 2f + u &= 14 \end{aligned}$$

where h, f and u represent the price of one hosomaki, futomaki and uramaki roll.

The values of h, f and u which satisfy the equations are the solution. The equations form a so called system of linear equations.

Definition

A **system of linear equations** (or **linear system**) is a set of linear equations in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

The a_{ij} are called the **coefficients** of the system and b_i are called the **right-hand side** of the system.
A **solution** of a linear system is any tuple of numbers that makes each equation a true statement. The set of all solutions of a system is called a **general solution** of the system.
Two systems with equal solution sets are called **equivalent**.
A system with no solution is called **inconsistent**.

Below is an example of a system of three linear equations in four variables x_1, x_2, x_3 and x_4.

$$\begin{aligned} 3x_1 - x_2 + 2x_3 + x_4 &= 4 \\ -x_1 + x_2 - x_3 + 5x_4 &= -2 \\ x_1 + 0.25x_4 &= 0 \end{aligned}$$

In this case $m = 3$, $n = 4$ and for example $b_1 = 4$, $a_{12} = -1$, $a_{33} = 0$.

6 – System of linear equations, Gaussian elimination

Some systems are easy to solve:

Exercise

Solve the following system.

$$\begin{aligned} x + y + z &= 10 \\ 3y + z &= 14 \\ 2z &= 10 \end{aligned}$$

Example

Solve the following system.

$$\begin{aligned} x + 3y + 2z &= 8 \\ 3y + 3z &= 15 \\ 2z &= 8 \end{aligned}$$

$$\begin{aligned} x + 3y + 2z &= 8 \\ 3y + 3z &= 15 \\ 2z &= 8 \end{aligned} \qquad \begin{aligned} x + 3 \cdot 1 + 2 \cdot 4 &= 8 & \longrightarrow \quad x &= -3 \\ 3y + 3 \cdot 4 &= 15 & \longrightarrow \quad y &= 1 \\ & & \longrightarrow \quad z &= 4 \end{aligned}$$

The last equation is solved for the value of the last variable. This value is then substituted into the previous equation to get the value of another variable and so on. This process is called **back substitution**.

Some operations with equations do not change the solution of the system. The simplest of these are called elementary operations. If such operations are performed on the rows of the augmented matrix of the system, they are called elementary row operations.

Definition

There are three types of **elementary row operations**:

- swapping two rows,
- multiplying a row by a nonzero number,
- adding a multiple of one row to another row.

7 – System of linear equations, Gaussian elimination

One way to find the solution of some system is to transform it into another simpler but equivalent system using elementary row operations. This method is called **Gaussian elimination**. It will be shown on the following 3 by 3 system.

Example

Solve the following system using Gaussian elimination.

$$\begin{aligned} x + 3y + z &= 2 \\ 2x + 8y + z &= 4 \\ 3x + y + 4z &= 0 \end{aligned}$$

The elementary row operatins are used to get zeros on the positions marked red.

$$\begin{aligned} x + 3y + z &= 2 && (R_1) \\ 2x + 8y + z &= 4 && (R_2) \\ 3x + y + 4z &= 0 && (R_3) \end{aligned}$$

To eliminate $2x$ in the second equation, the first equation is multiplied by 2 and subtracted from the second equation. The result is the following equivalent system:

$$\begin{aligned} x + 3y + z &= 2 \\ 2y - z &= 0 && (R_2 - 2R_1) \\ 3x + y + 4z &= 0 \end{aligned}$$

To eliminate $3x$ in the third equation, the first equation is multiplied by 3 and subtracted from the third equation.

$$\begin{aligned} x + 3y + z &= 2 \\ 2y - z &= 0 \\ -8y + z &= -6 && (R_3 - 3R_1) \end{aligned}$$

To eliminate $-8y$, the second equation is multiplied by -4 and subtracted from the third one.

$$\begin{aligned} x + 3y + z &= 2 \\ 2y - z &= 0 \\ -3z &= -6 && (R_3 + 4R_2) \end{aligned}$$

Back substitution is used to solve the system.

$$\begin{aligned} x + 3y + z &= 2 && x + 3 \cdot 1 + 1 \cdot 2 = 2 && \longrightarrow && x = -3 \\ 2y - z &= 0 && 2y - 1 \cdot 2 = 0 && \longrightarrow && y = 1 \\ -3z &= -6 && && \longrightarrow && z = 2 \end{aligned}$$

The solution is:

$$\begin{aligned} x &= -3 \\ y &= 1 \\ z &= 2 \end{aligned}$$

Verify that the solution is correct:

R_1: $(-3) + 3 \cdot 1 + 2 = 2$ ✓

R_2: $2 \cdot (-3) + 8 \cdot 1 + 2 = 4$ ✓

R_3: $3 \cdot (-3) + 1 + 4 \cdot 2 = 0$ ✓

8 – System of linear equations, Gaussian elimination

Gaussian elimination using matrices
There is no reason to write down the symbols for the variables (x, y, z, x_1 etc.) throughout the whole process of elimination since we only work with the numbers (coeficients and right-hand side) and only the numbers matter. The same process can be written more clearly by omitting the symbols.

So instead of
$$\begin{aligned} x + 3y + z &= 2 \\ 2x + 8y + z &= 4 \\ 3x + y + 4z &= 0 \end{aligned}$$
we write
$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 8 & 1 & 4 \\ 3 & 1 & 4 & 0 \end{pmatrix}.$$

An array of numbers such as this one is called a **matrix**. (See page 30 for a formal definition.) This particular matrix has three **rows** and four **columns**.

Definition

The matrix $\mathbf{A}$ containing the coefficients of a system is called its **coefficient matrix**. The matrix $\mathbf{A}|\mathbf{b}$ which contains in addition the right-hand side is called the **augmented matrix** of the system.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \qquad \mathbf{A}|\mathbf{b} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{pmatrix}$$

In our example,
$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 8 & 1 \\ 3 & 1 & 4 \end{pmatrix} \qquad \mathbf{A}|\mathbf{b} = \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 2 & 8 & 1 & 4 \\ 3 & 1 & 4 & 0 \end{array}\right)$$

The augmented matrix is ussually written with a line separating the right-hand side to emphasize it.

9 – System of linear equations, Gaussian elimination

Example

Solve the following system using Gaussian elimination.

$$\begin{aligned} x + 3y + z &= 2 \\ 2x + 8y + z &= 4 \\ 3x + y + 4z &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 2 & 8 & 1 & 4 \\ 3 & 1 & 4 & 0 \end{array}\right)_{R_2-2R_1} \to \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 2 & -1 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right)_{R_3-3R_1} \to$$

$$\to \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 2 & -1 & 0 \\ 0 & -8 & 1 & -6 \end{array}\right)_{R_3+4R_2} \to \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

To eliminate each column one of its nonzero coefficient is selected and used to eliminate coefficients below. Such a coefficient is called a **pivot**.

From now on pivots will be circled during elimination process as shown below.

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 2 & 8 & 1 & 4 \\ 3 & 1 & 4 & 0 \end{array}\right)_{R_2-2R_1} \to \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 2 & -1 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right)_{R_3-3R_1} \to$$

$$\to \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 2 & -1 & 0 \\ 0 & -8 & 1 & -6 \end{array}\right)_{R_3+4R_2} \to \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

The matrix resulting from elimination resembles a staircase and from now this also will be emphasized.

Back substitution is used to find the solution.

$$\begin{aligned} x + 3y + z &= 2 \\ 2y - z &= 0 \\ -3z &= -6 \end{aligned} \qquad \begin{aligned} x + 3\cdot 1 + 1\cdot 2 = 2 &\longrightarrow x = -3 \\ 2y - 1\cdot 2 = 0 &\longrightarrow y = 1 \\ &\longrightarrow z = 2 \end{aligned}$$

The system has one solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

Verify that the solution is correct:

R_1: $(-3) + 3\cdot 1 + 2 = 2$ ✓

R_2: $2\cdot(-3) + 8\cdot 1 + 2 = 4$ ✓

R_3: $3\cdot(-3) + 1 + 4\cdot 2 = 0$ ✓

10 – System of linear equations, Gaussian elimination

Exercise

Solve the following system using Gaussian elimination.

$$\begin{aligned} x + 3y + z &= 7 \\ -x - y + 2z &= 2 \\ 3x + y - 4z &= 0 \end{aligned}$$

11 – System of linear equations, Gaussian elimination

Example

Solve the following system using Gaussian elimination.

$$\begin{aligned}
13x_1 + 13x_2 + 13x_3 - 13x_4 &= 26\\
-x_1 - x_2 - 2x_3 + 2x_4 &= 2\\
2x_1 + 5x_2 - 3x_3 - x_4 &= -1\\
5x_1 + 7x_2 - 3x_3 - 2x_4 &= 2
\end{aligned}$$

$$\left(\begin{array}{cccc|c} 13 & 13 & 13 & -13 & 26\\ -1 & -1 & -2 & 2 & 2\\ 2 & 5 & -3 & -1 & -1\\ 5 & 7 & -3 & -2 & 2 \end{array}\right) \xrightarrow{R_1/13} \begin{array}{c} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2\\ -1 & -1 & -2 & 2 & 2\\ 2 & 5 & -3 & -1 & -1\\ 5 & 7 & -3 & -2 & 2 \end{array}\right) \end{array} \begin{array}{l} \\ R_2+R_1\\ R_3-2R_1\\ R_4-5R_1 \end{array} \to$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2\\ 0 & 0 & -1 & 1 & 4\\ 0 & 3 & -5 & 1 & -5\\ 0 & 2 & -8 & 3 & -8 \end{array}\right) \begin{array}{l} \\ R_2 \leftrightarrow R_3\\ \\ \\ \end{array} \to$$

A pivot cannot be zero so we will swap rows to get a nonzero pivot.

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2\\ 0 & 3 & -5 & 1 & -5\\ 0 & 0 & -1 & 1 & 4\\ 0 & 2 & -8 & 3 & -8 \end{array}\right) \begin{array}{l} \\ \\ \\ 3\,R_4 \end{array} \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2\\ 0 & 3 & -5 & 1 & -5\\ 0 & 0 & -1 & 1 & 4\\ 0 & 6 & -24 & 9 & -24 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4-2R_2 \end{array} \to$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2\\ 0 & 3 & -5 & 1 & -5\\ 0 & 0 & -1 & 1 & 4\\ 0 & 0 & -14 & 7 & -14 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4-14R_3 \end{array} \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2\\ 0 & 3 & -5 & 1 & -5\\ 0 & 0 & -1 & 1 & 4\\ 0 & 0 & 0 & -7 & -70 \end{array}\right)$$

Back substitution:

$$\begin{array}{rcl}
x_1 + x_2 + x_3 - x_4 = 2 & \quad x_1 + 1\cdot 5 + 1\cdot 6 - 1\cdot 10 = 2 & \longrightarrow x_1 = 1\\
3x_2 - 5x_3 + x_4 = -5 & \quad 3x_2 - 5\cdot 6 + 1\cdot 10 = -5 & \longrightarrow x_2 = 5\\
-x_3 + x_4 = 4 & \quad -x_3 + 1\cdot 10 = 4 & \longrightarrow x_3 = 6\\
-7x_4 = -70 & & \longrightarrow x_4 = 10
\end{array}$$

The solution is:

$$\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix} = \begin{pmatrix} 1\\ 5\\ 6\\ 10 \end{pmatrix}$$

Verify that the solution is correct:

R_1: $13\cdot 1 + 13\cdot 5 + 13\cdot 6 - 13\cdot 10 = 13 + 65 + 78 - 130 = 26$ ✓

R_2: $-1\cdot 1 - 5 - 2\cdot 6 + 2\cdot 10 = -1 - 5 - 12 + 20 = 2$ ✓

R_3: $2\cdot 1 + 5\cdot 5 - 3\cdot 6 - 10 = 2 + 25 - 18 - 10 = -1$ ✓

R_4: $5\cdot 1 + 7\cdot 5 - 3\cdot 6 - 2\cdot 10 = 5 + 35 - 18 - 20 = 2$ ✓

12 – System of linear equations, Gaussian elimination

Exercise

Solve the following system using Gaussian elimination.

$$\begin{aligned}
-2y-2z+8w&=2\\
x-3y-z+3w&=0\\
x-y+z+5w&=18\\
2x-8y+12w&=22\\
-x+y-z+5w&=2
\end{aligned}$$

13 – System of linear equations, Gaussian elimination

Example

Solve the following system using Gaussian elimination.

$$\begin{aligned} x + 4y + z &= 2 \\ 2x + 8y + z &= 4 \\ 3x + 12y + 4z &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 2 & 8 & 1 & 4 \\ 3 & 12 & 4 & 0 \end{array}\right)\begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -6 \end{array}\right) \rightarrow$$

In this case there is no pivot in second column. We choose a pivot from the third column and continue eliminating.

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -6 \end{array}\right)\begin{array}{l} \\ \\ R_3+R_2 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -6 \end{array}\right)$$

This matrix corresponds to the equations:

$$\begin{aligned} x + 4y + z &= 2 \\ - z &= 0 \\ 0z &= -6 \end{aligned}$$

There is no value of z such that $0z = -6$. Therefore the system has no solution.

System is inconsistent if after the elimination the column of right hand sides contains a pivot.

Below is an example of an inconsistent system with obvious inconsistency.

$$\begin{aligned} x + y + z &= 2 \\ x + y + z &= 3 \\ z &= 1 \end{aligned}$$

14 – System of linear equations, Gaussian elimination

Example

Solve the following system using Gaussian elimination.

$$\begin{aligned} x + 4y + z &= 2 \\ 2x + 8y + z &= 10 \\ 3x + 12y + 4z &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 2 & 8 & 1 & 10 \\ 3 & 12 & 4 & 0 \end{array}\right) \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array} \to \left(\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 1 & -6 \end{array}\right) \begin{array}{l} \\ \\ R_3+R_2 \end{array} \to \left(\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Back substitution:

$$\begin{aligned} x + 4y + z &= 2 \\ -z &= 6 \end{aligned} \qquad \begin{aligned} x + 4y + 1\cdot(-6) &= 2 \longrightarrow & x + 4y &= 8 \\ -z &= 6 \longrightarrow & z &= -6 \end{aligned}$$

For any chosen value of y there is a unique value of x that satisfies the equations. The system has infinitely many solutions.

For example $\begin{pmatrix} 0 \\ 2 \\ -6 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ -6 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \\ -6 \end{pmatrix}, \begin{pmatrix} 7.96 \\ 0.01 \\ -6 \end{pmatrix}$ etc.

If we choose t as the value of y, than we get the corresponding value of x from the first equation: $x + 4t = 8$, therefore $x = 8 - 4t$.
All solutions can be wrtitten in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8-4t \\ t \\ -6 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -6 \end{pmatrix} + t \cdot \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$$

where t is any real number.

A system can have many solutions because some equations in the system are redundant or because there are not "enough" equations. Below are some systems with obvious redundancy.

$$\begin{aligned} x + y + 3z &= 2 \\ 2x + 2y + 6z &= 4 \qquad (R_2 = 2R_1) \\ x - y + z &= 0 \end{aligned}$$

$$\begin{aligned} x + 2y + 3z &= 2 \\ x - y + 2z &= 3 \\ 2x + y + 5z &= 5 \qquad (R_3 = R_2 + R_1) \end{aligned}$$

15 – System of linear equations, Gaussian elimination

Exercise

Solve the following system using Gaussian elimination.

$$\begin{aligned} x + y + 2z &= 3 \\ 3x + 5y + 4z &= 13 \\ 2x + 3y + 3z &= 8 \end{aligned}$$

16 – System of linear equations, Gaussian elimination

Example

Solve the following system using Gaussian elimination.

$$\begin{aligned} x_1 + x_2 + x_3 \phantom{{}+x_4} + x_5 \phantom{{}+2x_6} &= 1 \\ 2x_1 + 2x_2 + 3x_3 + x_4 + x_5 + 2x_6 &= 6 \\ 3x_1 + 3x_2 + 4x_3 + x_4 + 2x_5 + 5x_6 &= 13 \\ x_1 + x_2 + 3x_3 + 2x_4 - x_5 + 4x_6 &= 9 \end{aligned}$$

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 2 & 3 & 1 & 1 & 2 & 6 \\ 3 & 3 & 4 & 1 & 2 & 5 & 13 \\ 1 & 1 & 3 & 2 & -1 & 4 & 9 \end{array}\right) \xrightarrow[\substack{R_3-3R_1 \\ R_4-R_1}]{R_2-2R_1} \left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 2 & 4 \\ 0 & 0 & 1 & 1 & -1 & 5 & 10 \\ 0 & 0 & 2 & 2 & -2 & 4 & 8 \end{array}\right) \xrightarrow[\substack{R_3-R_2 \\ R_4-2R_2}]{}$$

$$\rightarrow \left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The variables in the pivotal columns x_1, x_2 and x_5 are called **basic variables**. The other variables x_3 and x_4 are called **free variables**.

Back substitution:

$$\begin{aligned} x_1 + x_2 + x_3 + x_5 &= 1 & x_1 + r + (t-s) + t &= 1 & &\longrightarrow x_1 = 1 - r - 2t + s \\ & & & & &\longrightarrow x_2 = r \\ x_3 + x_4 - x_5 + 2x_6 &= 4 & x_3 + s - t + 2\cdot 2 &= 4 & &\longrightarrow x_3 = t - s \\ & & & & &\longrightarrow x_4 = s \\ & & & & &\longrightarrow x_5 = t \\ 3x_6 &= 6 & & & &\longrightarrow x_6 = 2 \end{aligned}$$

t, r and s are any real number.

The general solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1-r-2t+s \\ r \\ t-s \\ s \\ t \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -r \\ r \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2t \\ 0 \\ t \\ 0 \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} s \\ 0 \\ -s \\ s \\ 0 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} + r \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Verify that the solution is correct:

R_1: $(1-r-2t+s) + r + (t-s) + t = 1$ ✓

R_2: $2(1-r-2t+s) + 2r + 3(t-s) + s + t + 2\cdot 2 =$
$= 2 - 2r - 4t + 2s + 2r + 3t - 3s + s + t + 4 = 6$ ✓

R_3: $3(1-r-2t+s) + 3r + 4(t-s) + s + 2t + 5\cdot 2 =$
$= 3 - 3r - 6t + 3s + 3r + 4t - 4s + s + 2t + 10 = 13$ ✓

R_4: $(1-r-2t+s) + r + 3(t-s) + 2s - t + 4\cdot 2 =$
$= 1 - r - 2t + s + r + 3t - 3s + 2s - t + 8 = 9$ ✓

17 – System of linear equations, Gaussian elimination

Gaussian elimination is sequence of elementary row operations performed on the augmented matrix of a linear system to convert the matrix into so called row echelon form.

Definition

A matrix is said to be in **row echelon form** if each row except for the first starts with more zero then the row above it

$$\begin{pmatrix} \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast \\ 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\ast$ nonzero pivot
$\ast$ any number
basic (pivot) columns

Columns containing a pivot are called **basic columns** or **pivot columns**.

Every matrix can be transformed into a matrix in row echelon form. Because the process is flexible, the numbers in the echelon form are not uniquely determined. Nevertheless, it can be proven that the ”shape” of the echelon form and in particular the position and number of pivots is uniquely determined by the matrix.

Definition

The **rank** of a matrix $\mathbf{A}$ is the number of nonzero rows in the matrix in row echelon form obtained from matrix $\mathbf{A}$ by Gaussian elimination. It is denoted by $\operatorname{rank}(\mathbf{A})$.

There are three possibilities for the number of solutions of a linear system:

- A system has **no solution** if one of the pivots sits in the right-hand side column.

For example:
$$\left(\begin{array}{cccccc|c} \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ 0 & 0 & \ast & \ast & \ast & \ast & \ast \\ 0 & 0 & 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & 0 & 0 & 0 & \ast \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- A system has **one solution** if none of the pivots sit in the right-hand side column and the number of variables is equal to the number of nonzero rows.

For example:
$$\left(\begin{array}{cccc|c} \ast & \ast & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast & \ast \\ 0 & 0 & \ast & \ast & \ast \\ 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- A system has **infinitely many solutions** if none of the pivots sit in the right-hand side column and the number of variables is less than the number of nonzero rows.

For example:
$$\left(\begin{array}{cccccc|c} \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ 0 & 0 & \ast & \ast & \ast & \ast & \ast \\ 0 & 0 & 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

18 – System of linear equations, Gaussian elimination

Exercise

Solve the following system using Gaussian elimination.

$$\begin{aligned} 2x + 4y - 6z + 2w &= 2 \\ 5x + 10y - 15z - 4w &= 1 \\ -3x - 6y + 9z - 2w &= 6 \end{aligned}$$

19 – System of linear equations, Gaussian elimination

Exercise

Solve the following system using Gaussian elimination.

$$\begin{aligned} x + 2y + z + 2w &= 0 \\ 2x + 4y + 4z + 4w &= 4 \\ 3x + 6y + 5z + 6w &= 4 \\ -x - 2y + z - 2w &= 4 \end{aligned}$$

20 – System of linear equations, Gaussian elimination

Example

Find the price of those rolls from sushi bar.

$$\begin{aligned} h+\ 5f + 2u &= 24 \\ 3h + 4f + 2u &= 23 \\ 4h + 2f + \ u &= 14 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 5 & 2 & 24 \\ 3 & 4 & 2 & 23 \\ 4 & 2 & 1 & 14 \end{array}\right)\begin{array}{l} \\ \scriptstyle R_2-3R_1 \\ \scriptstyle R_3-4R_1 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & 2 & 24 \\ 0 & -11 & -4 & -49 \\ 0 & -18 & -7 & -82 \end{array}\right)_{11R_3-18R_2} \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & 2 & 24 \\ 0 & -11 & -4 & -49 \\ 0 & 0 & -5 & -20 \end{array}\right)$$

$$\begin{aligned} h + 5f + 2u &= 24 \\ -11f - 4u &= -49 \\ -5u &= -20 \end{aligned} \qquad \begin{aligned} h + 15 + 8 &= 24 &\longrightarrow\quad h &= 1 \\ -11f - 16 &= -49 &\longrightarrow\quad f &= 3 \\ & &\longrightarrow\quad u &= 4 \end{aligned}$$

One hosomaki roll costs 1 coin, one futomaki roll costs 3 coins and one uramaki roll costs 4 coins.

21 – System of linear equations, Gauss-Jordan elimination

Gauss-Jordan elimination is a modification of Gaussian elimination with two differences. All pivots are forced to be 1 and all entries not only below but also above them are eliminated.

Example

Solve the following system using Gauss-Jordan elimination.

$$\begin{aligned} x + 5y + 2z &= -2 \\ -2x - 8y + 4z &= 14 \\ x + 8y + 9z &= 3 \end{aligned}$$

We perform elimination to transform the coefficient matrix in row echelon form.

$$\left(\begin{array}{ccc|c} 1 & 5 & 2 & -2 \\ -2 & -8 & 4 & 14 \\ 1 & 8 & 9 & 3 \end{array}\right) \begin{array}{l} \\ R_2-2R_1 \\ R_3-R_1 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & 2 & -2 \\ 0 & 2 & 8 & 10 \\ 0 & 3 & 7 & 5 \end{array}\right) R_2/2 \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 5 & 2 & -2 \\ 0 & 1 & 4 & 5 \\ 0 & 3 & 7 & 5 \end{array}\right)_{R_3-3R_2} \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & 2 & -2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & -5 & -10 \end{array}\right)_{R_3/-5} \rightarrow$$

The matrix is now in row echelon form. But instead of stopping we continue and eliminate all entries above pivots.

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 5 & 2 & -2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} R_1-2R_3 \\ R_2-4R_3 \\ \\ \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} R_1-5R_2 \\ \\ \\ \end{array} \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

The solution is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ 2 \end{pmatrix}$$

Definition

A matrix is said to be in **reduced row echelon form** if

- it is in row echelon form,
- all pivots are 1, and
- all entries above each pivot are zero.

$$\begin{pmatrix} 1 & 0 & * & 0 & * & * & 0 & * \\ 0 & 1 & * & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad * \text{ any number}$$

If a matrix **A** is transformed into a matrix in reduced row echelon form, not only the shape of it is uniquely determined by **A** but also all the entries.

There is a unique matrix in row echelon form associated with any matrix **A**. It is usually denoted by rref(**A**).

22 – System of linear equations, Gauss-Jordan elimination

Example

Solve the following system using Gauss-Jordan elimination.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 3x_4 + 3x_5 + 2x_6 &= 5 \\ 2x_1 + 4x_2 - 3x_3 + 4x_4 + 7x_5 + 10x_6 &= 21 \\ 3x_1 + 6x_2 - 3x_3 + 6x_4 + 12x_5 + 14x_6 &= 29 \\ x_1 + 2x_2 - 3x_3 + 4x_4 + 4x_5 &= 2 \end{aligned}$$

$$\left(\begin{array}{cccccc|c} 1 & 2 & -3 & 3 & 3 & 2 & 5 \\ 2 & 4 & -3 & 4 & 7 & 10 & 21 \\ 3 & 6 & -3 & 6 & 12 & 14 & 29 \\ 1 & 2 & -3 & 4 & 4 & 0 & 2 \end{array}\right) \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \\ R_4-R_1 \end{array} \to \left(\begin{array}{cccccc|c} 1 & 2 & -3 & 3 & 3 & 2 & 5 \\ 0 & 0 & 3 & -2 & 1 & 6 & 11 \\ 0 & 0 & 6 & -3 & 3 & 8 & 14 \\ 0 & 0 & 0 & 1 & 1 & -2 & -3 \end{array}\right) \begin{array}{l} \\ \\ R_3-2R_2 \\ \\ \end{array} \to \left(\begin{array}{cccccc|c} 1 & 2 & -3 & 3 & 3 & 2 & 5 \\ 0 & 0 & 3 & -2 & 1 & 6 & 11 \\ 0 & 0 & 0 & 1 & 1 & -4 & -8 \\ 0 & 0 & 0 & 1 & 1 & -2 & -3 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4-R_3 \end{array} \to \left(\begin{array}{cccccc|c} 1 & 2 & -3 & 3 & 3 & 2 & 5 \\ 0 & 0 & 3 & -2 & 1 & 6 & 11 \\ 0 & 0 & 0 & 1 & 1 & -4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 2 & 5 \end{array}\right) \begin{array}{l} R_1-R_4 \\ R_2-3R_4 \\ R_3+2R_4 \\ \\ \end{array} \to$$

At this point, the forward part of Gaussian elimination is finished. The coefficient matrix is in echelon form. The following additional row operations are performed to transform the matrix to reduced echelon form.

$$\to \left(\begin{array}{cccccc|c} 1 & 2 & -3 & 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & -2 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 5 \end{array}\right) \begin{array}{l} R_1-3R_3 \\ R_2+2R_3 \\ \\ \\ \end{array} \to \left(\begin{array}{cccccc|c} 1 & 2 & -3 & 0 & 0 & 0 & -6 \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 5 \end{array}\right) \begin{array}{l} R_1+R_2 \\ \\ \\ \\ \end{array} \to \left(\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 3 & 0 & -6 \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 5 \end{array}\right) \begin{array}{l} \\ R_2/3 \\ \\ R_4/2 \end{array} \to \left(\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 3 & 0 & -6 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5/2 \end{array}\right)$$

The solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -6-2s-3t \\ s \\ -t \\ 2-t \\ t \\ 5/2 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 0 \\ 2 \\ 0 \\ 5/2 \end{pmatrix} + s \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

Verify that the solution is correct:

R_1: $(-6-2s-3t)+2s-3(-t)+3(2-t)+3t+2\cdot(5/2)=5$ ✓

R_2: $2(-6-2s-3t)+4s-3(-t)+4(2-t)+7t+10\cdot(5/2)=21$ ✓

R_3: $3(-6-2s-3t)+6s-3(-t)+6(2-t)+12t+14\cdot(5/2)=29$ ✓

R_4: $(-6-2s-3t)+2s-3(-t)+4(2-t)+4t=2$ ✓

23 – System of linear equations, Gauss-Jordan elimination

Exercise

Solve the following system using Gauss-Jordan elimination.

$$
\begin{aligned}
x_2 - 4x_3 + x_4 \phantom{{}-13x_5} &= 4 \\
2x_1 + x_2 + 2x_3 \phantom{{}+x_4} - 8x_5 &= 3 \\
3x_1 + 2x_2 + x_3 + x_4 - 13x_5 &= 7
\end{aligned}
$$

24 – System of linear equations, Gauss-Jordan elimination

Exercise

Solve the following system using Gauss-Jordan elimination.

$$
\begin{aligned}
5x + 10y + 5z &= 1005 \\
100x - y + 4z &= 0 \\
2x + 10z &= 2
\end{aligned}
$$

25 – System of linear equations, Gauss-Jordan elimination

Example

Use the Gauss-Jordan elimination to solve the following three systems with the same coefficient matrix at the same time.

$$\begin{aligned} x + 3y + z &= 7 \\ 2x + y + z &= 0 \\ 3x + y + 4z &= 9 \end{aligned} \qquad \begin{aligned} x + 3y + z &= 4 \\ 2x + y + z &= 2 \\ 3x + y + 4z &= 5 \end{aligned} \qquad \begin{aligned} x + 3y + z &= 8 \\ 2x + y + z &= 6 \\ 3x + y + 4z &= 8 \end{aligned}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 7 & 4 & 8 \\ 2 & 1 & 1 & 0 & 2 & 6 \\ 3 & 1 & 4 & 9 & 5 & 8 \end{array}\right) \begin{matrix} \\ R_2-2R_1 \\ R_3-3R_1 \end{matrix} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 7 & 4 & 8 \\ 0 & -5 & -1 & -14 & -6 & -10 \\ 0 & -8 & 1 & -12 & -7 & -16 \end{array}\right)_{5R_3-8R_2} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 7 & 4 & 8 \\ 0 & -5 & -1 & -14 & -6 & -10 \\ 0 & 0 & 13 & 52 & 13 & 0 \end{array}\right)_{R_3/13} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 7 & 4 & 8 \\ 0 & -5 & -1 & -14 & -6 & -10 \\ 0 & 0 & 1 & 4 & 1 & 0 \end{array}\right) \begin{matrix} R_1-R_3 \\ R_2+R_3 \\ \\ \end{matrix} \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 3 & 3 & 8 \\ 0 & -5 & 0 & -10 & -5 & -10 \\ 0 & 0 & 1 & 4 & 1 & 0 \end{array}\right) R_2/-5 \rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 3 & 3 & 8 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 4 & 1 & 0 \end{array}\right)^{R_1-3R_2} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 4 & 1 & 0 \end{array}\right)$$

The solution of the first system is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}$$

The solution of the second system is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The solution of the third system is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Verify that the solution is correct:

R_1: $-3+6+4=7$ ✓
R_2: $-6+2+4=0$ ✓
R_3: $-9+2+16=9$ ✓

Verify that the solution is correct:

R_1: $0+3+1=4$ ✓
R_2: $0+1+1=2$ ✓
R_3: $0+1+4=5$ ✓

Verify that the solution is correct:

R_1: $2+6+0=8$ ✓
R_2: $4+2+0=6$ ✓
R_3: $6+2+0=8$ ✓

26 – System of linear equations, homogeneous and nonhomogeneous systems

Definition

A system of linear equations with the right-hand side consisting entirely of zeros is said to be **homogeneous**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned},$$

A system with at least one nonzero number on the right-hand side is called **nonhomogeneous**.

For example:
An nonhomogeneous system

$$\begin{aligned} x + 4y - z + 2w &= 2 \\ 5x + y - z - 4w &= 1 \\ -3x - 6y + z - 2w &= 0 \end{aligned}$$

and the associated homogenous system

$$\begin{aligned} x + 4y - z + 2w &= 0 \\ 5x + y - z - 4w &= 0 \\ -3x - 6y + z - 2w &= 0 \end{aligned}$$

A homogeneous system has always at least one solution, the **trivial solution** consisting of all zeros.

Exercise

Solve the following system.

$$\begin{aligned} -x + y - z &= 0 \\ 3x - y - z &= 0 \\ 2x + y - 3z &= 0 \end{aligned}$$

27 – System of linear equations, homogeneous and nonhomogeneous systems

There is a close relation between the solution of a nonhomogeneous system and the solution of the associated homogeneous one.

Example

Find and compare solutions to the following systems.

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 + x_5 &= 0 \\ -x_1 + 2x_2 + x_3 - x_5 &= 0 \\ -x_1 - x_2 + x_3 + x_4 + 5x_5 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 - x_3 + x_4 + x_5 &= 9 \\ -x_1 + 2x_2 + x_3 - x_5 &= 5 \\ -x_1 - x_2 + x_3 + x_4 + 5x_5 &= 1 \end{aligned} \qquad \begin{aligned} x_1 + x_2 - x_3 + x_4 + x_5 &= -6 \\ -x_1 + 2x_2 + x_3 - x_5 &= 6 \\ -x_1 - x_2 + x_3 + x_4 + 5x_5 &= 0 \end{aligned}$$

The elimination will be performed on all three systems at the same time.

$$\left(\begin{array}{rrrrr|rrr} 1 & 1 & -1 & 1 & 1 & 0 & 9 & -6 \\ -1 & 2 & 1 & 0 & -1 & 0 & 5 & 6 \\ -1 & -1 & 1 & 1 & 5 & 0 & 1 & 0 \end{array}\right)\begin{array}{l} \\ R_2+R_1 \\ R_3+R_1 \end{array} \to \left(\begin{array}{rrrrr|rrr} 1 & 1 & -1 & 1 & 1 & 0 & 9 & -6 \\ 0 & 3 & 0 & 1 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 2 & 6 & 0 & 10 & -6 \end{array}\right)\begin{array}{l} \\ \\ R_3/2 \end{array} \to \left(\begin{array}{rrrrr|rrr} 1 & 1 & -1 & 1 & 1 & 0 & 9 & -6 \\ 0 & 3 & 0 & 1 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 & -3 \end{array}\right)\begin{array}{l} R_1-R_3 \\ R_2-R_3 \\ \\ \end{array} \to$$

$$\to \left(\begin{array}{rrrrr|rrr} 1 & 1 & -1 & 0 & -2 & 0 & 4 & -3 \\ 0 & 3 & 0 & 0 & -3 & 0 & 9 & 3 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 & -3 \end{array}\right)\begin{array}{l} \\ R_2/3 \\ \\ \end{array} \to \left(\begin{array}{rrrrr|rrr} 1 & 1 & -1 & 0 & -2 & 0 & 4 & -3 \\ 0 & 1 & 0 & 0 & -1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 & -3 \end{array}\right)\begin{array}{l} R_1-R_2 \\ \\ \\ \end{array} \to \left(\begin{array}{rrrrr|rrr} 1 & 0 & -1 & 0 & -1 & 0 & 1 & -4 \\ 0 & 1 & 0 & 0 & -1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 & -3 \end{array}\right)$$

$$\begin{aligned} x_1 - x_3 - x_5 &= 0 \\ x_2 - x_5 &= 0 \\ \\ x_4 + 3x_5 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= t + s \\ x_2 &= t \\ x_3 &= s \\ x_4 &= -3t \\ x_5 &= t \end{aligned} \qquad \begin{aligned} x_1 - x_3 - x_5 &= 1 \\ x_2 - x_5 &= 3 \\ \\ x_4 + 3x_5 &= 5 \end{aligned} \qquad \begin{aligned} x_1 &= 1 + t + s \\ x_2 &= 3 + t \\ x_3 &= s \\ x_4 &= 5 - 3t \\ x_5 &= t \end{aligned} \qquad \begin{aligned} x_1 - x_3 - x_5 &= -4 \\ x_2 - x_5 &= 1 \\ \\ x_4 + 3x_5 &= -3 \end{aligned} \qquad \begin{aligned} x_1 &= -4 + t + s \\ x_2 &= 1 + t \\ x_3 &= s \\ x_4 &= -3 - 3t \\ x_5 &= t \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 0 \\ -3 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \\ -3 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 0 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \\ -3 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

A general solution of the nonhomogeneous system is the sum of a so called **particular solution** (the green part) and the solution of the associated homogeneous system (the blue part).

28 – System of linear equations, homogeneous and nonhomogeneous systems

Exercise

Find the solution to the following systems.

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 + x_5 &= 0 \\ -x_1 + 2x_2 + x_3 - x_5 &= 0 \\ -x_1 - x_2 + x_3 + x_4 + 5x_5 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 - x_3 + x_4 + x_5 &= 1 \\ -x_1 + 2x_2 + x_3 - x_5 &= 1 \\ -x_1 - x_2 + x_3 + x_4 + 5x_5 &= 2 \end{aligned}$$

Worksheets for Mathematics I

Matrix algebra

Jana Bělohlávková

30 – Matrix algebra

Definition

An array of numbers (real or complex) is called a **matrix**.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

The number a_{ij} in the ith row and jth column is called an **entry** of the matrix. It can be also denoted $(\mathbf{A})_{ij}$.
The **size** of the matrix is denoted $m \times n$ (pronounced "m by n").
The entries $a_{11}, a_{22}, a_{33}, \ldots$ make up the **main diagonal**.

For example (main diagonals are blue):

$$\begin{pmatrix} 1 & 5 & 7 \\ 1 & 4 & 3 \\ 3 & 7 & 8 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 5 & 1 & 1 \\ 0 & 4 & 1 & 4 & 0 \\ 1 & 0 & 5 & 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 5 & 1 \\ 0 & 4 & 1 & 4 \\ 1 & 0 & 5 & 2 \\ 1 & 1 & 5 & 1 \\ 1 & 4 & 0 & 3 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

Definition

A matrix is called a **square matrix** when it has the same number of rows and columns.
Otherwise the matrix is said to be **rectangular**.

Definition

Matrices with all entries equal to zero are called **zero matrices** and are denoted **O**.
Square matrices with ones on the main diagonal and zeros everywhere else are called **identity matrices** and are denoted **I**.

For example:

$$\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition

A square matrix is called **lower triangular** if all the entries above the main diagonal are zero.
A square matrix is called **upper triangular** if all the entries below the main diagonal are zero.

For example:

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 7 & 1 \end{pmatrix} \qquad \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 2/7 & 8 \end{pmatrix} \qquad \mathbf{U} = \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 6 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

31 – Matrix algebra, matrix operations

Definition

The **product of a number** k **and a matrix A** is defined to be the matrix obtained by multiplying each entry of **A** by k.

$$(k \cdot \mathbf{A})_{ij} = k \cdot a_{ij}$$

For example:

$$3\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 6 & 0 \\ 9 & 24 \end{pmatrix}$$

Definition

The **sum of two matrices** $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{m\times n}$ is defined to be the $m \times n$ matrix obtained by adding corresponding entries.

$$(\mathbf{A}+\mathbf{B})_{ij} = a_{ij} + b_{ij}$$

For example:

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 3 & 5 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 5 \\ 6 & 12 \end{pmatrix}$$

Addition laws:

$$\begin{aligned} \mathbf{A}+\mathbf{B} &= \mathbf{B}+\mathbf{A} && \text{(commutative law)} \\ k\cdot(\mathbf{A}+\mathbf{B}) &= k\cdot\mathbf{A}+k\cdot\mathbf{B} && \text{(distributive law)} \\ \mathbf{A}+(\mathbf{B}+\mathbf{C}) &= (\mathbf{A}+\mathbf{B})+\mathbf{C} && \text{(associative law)} \end{aligned}$$

Definition

The **transpose** of $\mathbf{A}_{m\times n}$ is defined to be the $n \times m$ matrix $\mathbf{A}^\mathsf{T}$ obtained by flipping **A** over its main diagonal.

$$(\mathbf{A}^\mathsf{T})_{ij} = a_{ji}$$

For example:

$$\begin{pmatrix} 1 & 3 & 7 \\ 2 & 8 & 1 \end{pmatrix}^\mathsf{T} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \\ 7 & 1 \end{pmatrix}$$

It is evident that for all matrices, $(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}$.

Exercise

Let **A** and **B** be the matrices as follows. Determine the matrices $3\cdot\mathbf{A}+2\cdot\mathbf{B}$ and $\mathbf{A}^\mathsf{T}$.

$$\mathbf{A} = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 7 & -1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 5 \\ -1 & -2 & 100 \end{pmatrix}$$

32 – Matrix algebra, matrix operations

Definition

The **product of two matrices** $\mathbf{A}_{m\times p}$ and $\mathbf{B}_{p\times n}$ is defined to be the $m\times n$ matrix whose ijth entry is obtained by "multiplying" ith row of $\mathbf{A}$ with jth column of $\mathbf{B}$ as follows:

$$(\mathbf{A}\cdot\mathbf{B})_{ij} = \sum_{k=1}^{p} a_{ik}\cdot b_{kj}$$

Exercise

Evaluate the following:

$$\begin{pmatrix} 2 & 1 & 2 & -6 & -5 \\ 0 & 1 & -4 & 1 & 0 \\ 3 & 1 & 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

For example:

a) $$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 1\cdot 4 + 2\cdot 4 + 3\cdot 4 \\ 0\cdot 4 + 7\cdot 4 + 0\cdot 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 28 \end{pmatrix}$$

b) $$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1\cdot 5 + 2\cdot 5 + 3\cdot 5 \\ 0\cdot 5 + 7\cdot 5 + 0\cdot 5 \end{pmatrix} = \begin{pmatrix} 30 \\ 35 \end{pmatrix}$$

c) $$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 0 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 4 & 5 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 24 & 30 \\ 38 & 35 \end{pmatrix}$$

33 – Matrix algebra, matrix operations

Exercise

Evaluate the following:

a) $\begin{pmatrix} 3 & 2 \\ 0 & 6 \\ 5 & 1 \\ 4 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 6 & -1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ 2 & 4 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 2 \\ 0 & 6 \\ 5 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}$

34 – Matrix algebra, matrix operations

Exercise

Find $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ for the following matrices.

$$\mathbf{A} = \begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \end{pmatrix}$$

Exercise

Find $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ for the following matrices.

$$\mathbf{A} = \begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Exercise

Find $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ for the following matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 7 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

For most matrices

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$

even when both products exist and have the same shape.
Matrix multiplication is not comutative.

35 – Matrix algebra, matrix operations

Exercise

Find $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}$ and $\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$ for the following matrices.

$$\mathbf{A}=\begin{pmatrix}2 & 1\\ 6 & -1\end{pmatrix}\qquad \mathbf{B}=\begin{pmatrix}2 & 0\\ 1 & 3\end{pmatrix}\qquad \mathbf{C}=\begin{pmatrix}5 & 4\\ -1 & 2\end{pmatrix}$$

Multiplication laws:

$$\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})=(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}\qquad\text{(associative law)}$$
$$(\mathbf{A}+\mathbf{B})\cdot\mathbf{C}=\mathbf{A}\cdot\mathbf{C}+\mathbf{B}\cdot\mathbf{C}\qquad\text{(distributive law)}$$
$$\mathbf{C}\cdot(\mathbf{A}+\mathbf{B})=\mathbf{C}\cdot\mathbf{A}+\mathbf{C}\cdot\mathbf{B}\qquad\text{(distributive law)}$$

Because $\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})=(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}$, there is no need to write parentheses and we can simply write $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$.

$$\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})=(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}=\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$$

Similarly, can write $\mathbf{A}^3$ instead of $\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}$, etc.

Definition

For any positive integer k **the kth power of the square matrix $\mathbf{A}$** is defined as the product of k matrices $\mathbf{A}$.

$$\mathbf{A}^k=\underbrace{\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\dots\mathbf{A}}_{k\text{ times}}$$

A matrix to the *zero*th power is defined to be the identity matrix of the same size $\mathbf{A}^0=\mathbf{I}$.

36 – Matrix algebra, matrix inverse

Example

Evaluate the following:

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Example

Evaluate the following:

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} 3 & 0 & -2 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Example

Evaluate the following:

$$\begin{pmatrix} 3 & 0 & -2 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 7 & 7 \\ 9 & 9 \\ 10 & 10 \end{pmatrix}$$

Definition

A square matrix $\mathbf{A}$ is called **invertible** if there exists a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$$

The matrix $\mathbf{A}^{-1}$ is called the **inverse** of $\mathbf{A}$.
A square matrix which is not invertible is called **singular**.

Although not all matrices are invertible, when an inverse exists, it is unique.

37 – Matrix algebra, matrix inverse

Gauss-Jordan elimination can be used to compute an inverse.

Example

Find the inverse of the matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix}$$

$$(\,\mathbf{A}\mid\mathbf{I}\,) = \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow[R_3-R_2]{R_1-R_2}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \xrightarrow[R_2-R_3]{R_1-2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) = (\,\mathbf{I}\mid\mathbf{A}^{-1})$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & 0 & -2 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Verify that the solution is correct:

$$\mathbf{A}\cdot\mathbf{A}^{-1} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} 3 & 0 & -2 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1}\cdot\mathbf{A} = \begin{pmatrix} 3 & 0 & -2 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

Find the inverse of the matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$(\,\mathbf{A}\mid\mathbf{I}\,) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3-R_1]{} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow[R_3-R_2]{}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ {\color{red}0} & {\color{red}0} & {\color{red}0} & -1 & -1 & 1 \end{array}\right)$$

The matrix $\mathbf{A}$ cannot be reduced to an identity matrix because a zero row emerged during elimination. Therefore $\mathbf{A}$ is singular.

A $n \times n$ matrix is invertible if and only if its rank is n.

For two invertible matrices $\mathbf{A}$ and $\mathbf{B}$, the following properties hold.

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- The product $\mathbf{A}\cdot\mathbf{B}$ is also invertible.
- $(\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$ (the reverse order law for inversion)

38 – Matrix algebra, matrix inverse

Example

Find the inverse of the matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 8 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

$$(\,\mathbf{A} \mid \mathbf{I}\,) = \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 8 & 1 & 0 & 1 & 0 \\ 3 & 1 & 4 & 0 & 0 & 1 \end{array}\right)\begin{matrix} \\ \scriptstyle R_2-2R_1 \\ \scriptstyle R_3-3R_1 \end{matrix} \to \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & -8 & 1 & -3 & 0 & 1 \end{array}\right)\begin{matrix} \\ \\ \scriptstyle R_3+4R_2 \end{matrix} \to \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 0 & -3 & -11 & 4 & 1 \end{array}\right)\begin{matrix} \scriptstyle 3R_1 \\ \scriptstyle 3R_2 \\ \\ \end{matrix} \to \left(\begin{array}{ccc|ccc} 3 & 9 & 3 & 3 & 0 & 0 \\ 0 & 6 & -3 & -6 & 3 & 0 \\ 0 & 0 & -3 & -11 & 4 & 1 \end{array}\right)\begin{matrix} \scriptstyle R_1+R_3 \\ \scriptstyle R_2-R_3 \\ \\ \end{matrix} \to$$

$$\to \left(\begin{array}{ccc|ccc} 3 & 9 & 0 & -8 & 4 & 1 \\ 0 & 6 & 0 & 5 & -1 & -1 \\ 0 & 0 & -3 & -11 & 4 & 1 \end{array}\right)\begin{matrix} \scriptstyle 2R_1 \\ \scriptstyle 3R_2 \\ \\ \end{matrix} \to \left(\begin{array}{ccc|ccc} 6 & 18 & 0 & -16 & 8 & 2 \\ 0 & 18 & 0 & 15 & -3 & -3 \\ 0 & 0 & -3 & -11 & 4 & 1 \end{array}\right)\begin{matrix} \scriptstyle R_1-R_2 \\ \\ \\ \end{matrix} \to \left(\begin{array}{ccc|ccc} 6 & 0 & 0 & -31 & 11 & 5 \\ 0 & 18 & 0 & 15 & -3 & -3 \\ 0 & 0 & -3 & -11 & 4 & 1 \end{array}\right)\begin{matrix} \scriptstyle R_1/6 \\ \scriptstyle R_2/18 \\ \scriptstyle R_3/(-3) \end{matrix} \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -31/6 & 11/6 & 5/6 \\ 0 & 1 & 0 & 5/6 & -1/6 & -1/6 \\ 0 & 0 & 1 & 11/3 & -4/3 & -1/3 \end{array}\right) = (\,\mathbf{I} \mid \mathbf{A}^{-1}\,)$$

$$\mathbf{A}^{-1} = 1/6 \begin{pmatrix} -31 & 11 & 5 \\ 5 & -1 & -1 \\ 22 & -8 & -2 \end{pmatrix}$$

Verify that the solution is correct:

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 8 & 1 \\ 3 & 1 & 4 \end{pmatrix} 1/6 \begin{pmatrix} -31 & 11 & 5 \\ 5 & -1 & -1 \\ 22 & -8 & -2 \end{pmatrix} = 1/6 \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} \cdot \mathbf{A} = 1/6 \begin{pmatrix} -31 & 11 & 5 \\ 5 & -1 & -1 \\ 22 & -8 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 8 & 1 \\ 3 & 1 & 4 \end{pmatrix} = 1/6 \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

39 – Matrix algebra, matrix inverse

Exercise

Find the inverse of the matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} -3 & -1 & 3 \\ 1 & -3 & -1 \\ -2 & 5 & 3 \end{pmatrix}$$

40 – Matrix algebra, matrix inverse

Exercise

Find the inverse of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 3 & 5 & 2 \\ -1 & -3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

41 – Matrix algebra, matrix inverse

Exercise

Find the inverse of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

42 – Matrix algebra, matrix inverse

Exercise

Find the inverse of the matrix **A** and **B**.

a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ b) $\mathbf{B} = \begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix}$

43 – Matrix algebra, matrix equations

A system of linear equations can be written as a matrix equation.

A system of linear equations:
$$\begin{aligned} x_1 + 3x_2 + x_3 &= 2 \\ 2x_1 + 8x_2 + x_3 &= 4 \\ 3x_1 + x_2 + 4x_3 &= 0 \end{aligned}$$

Its corresponding matrix equation:
$$\underbrace{\begin{pmatrix} 1 & 3 & 1 \\ 2 & 8 & 1 \\ 3 & 1 & 4 \end{pmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}}_{\mathbf{b}}$$

The matrix equation $\mathbf{A} \cdot \mathbf{X} = \mathbf{B}$ can be solved if the matrix $\mathbf{A}$ is invertible:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{X} &= \mathbf{B} \\ \mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{X} &= \mathbf{A}^{-1} \cdot \mathbf{B} \qquad \text{multiply by } \mathbf{A}^{-1} \text{ from the left} \\ \underbrace{\mathbf{A}^{-1} \cdot \mathbf{A}}_{\mathbf{I}} \cdot \mathbf{X} &= \mathbf{A}^{-1} \cdot \mathbf{B} \\ \mathbf{I} \cdot \mathbf{X} &= \mathbf{A}^{-1} \cdot \mathbf{B} \\ \mathbf{X} &= \mathbf{A}^{-1} \cdot \mathbf{B} \end{aligned}$$

The matrix equation $\mathbf{X} \cdot \mathbf{A} = \mathbf{B}$ can be solved if the matrix $\mathbf{A}$ is invertible:

$$\begin{aligned} \mathbf{X} \cdot \mathbf{A} &= \mathbf{B} \\ \mathbf{X} \cdot \mathbf{A} \cdot \mathbf{A}^{-1} &= \mathbf{B} \cdot \mathbf{A}^{-1} \qquad \text{multiply by } \mathbf{A}^{-1} \text{ from the right} \\ \mathbf{X} \cdot \underbrace{\mathbf{A} \cdot \mathbf{A}^{-1}}_{\mathbf{I}} &= \mathbf{B} \cdot \mathbf{A}^{-1} \\ \mathbf{X} \cdot \mathbf{I} &= \mathbf{B} \cdot \mathbf{A}^{-1} \\ \mathbf{X} &= \mathbf{B} \cdot \mathbf{A}^{-1} \end{aligned}$$

Example

Solve the following matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$.

$$\underbrace{\begin{pmatrix} 1 & 3 & 1 \\ 2 & 8 & 1 \\ 3 & 1 & 4 \end{pmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}}_{\mathbf{b}}$$

Matrix $\mathbf{A}$ has the inverse $\mathbf{A}^{-1} = {}^1/_6 \begin{pmatrix} -31 & 11 & 5 \\ 5 & -1 & -1 \\ 22 & -8 & -2 \end{pmatrix}$, see worksheet 38.

$$\begin{aligned} \mathbf{A} \cdot \mathbf{x} &= \mathbf{b} \\ \mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} &= \mathbf{A}^{-1} \cdot \mathbf{b} \\ \mathbf{x} &= \mathbf{A}^{-1} \cdot \mathbf{b} \end{aligned}$$

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} = {}^1/_6 \begin{pmatrix} -31 & 11 & 5 \\ 5 & -1 & -1 \\ 22 & -8 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = {}^1/_6 \begin{pmatrix} -18 \\ 6 \\ 12 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

Verify that the solution is correct:

$$\mathbf{A} \cdot \mathbf{x} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 8 & 1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \mathbf{b} \quad \checkmark$$

44 – Matrix algebra, matrix equations

Example

Solve the following matrix equation for the unknown matrix $\mathbf{X}$.

$$\mathbf{X}\cdot\mathbf{F}\cdot\mathbf{G}=\mathbf{B} \qquad \mathbf{F}=\begin{pmatrix}0&-1\\2&5\end{pmatrix} \qquad \mathbf{G}=\begin{pmatrix}2&-7\\-1&4\end{pmatrix} \qquad \mathbf{B}=\begin{pmatrix}-1&8\\-1&12\end{pmatrix}$$

$$(\,\mathbf{F}\mid\mathbf{I}\,)=\left(\begin{array}{cc|cc}0&-1&1&0\\2&5&0&1\end{array}\right)\xrightarrow[R_2\leftrightarrow R_1]{}\left(\begin{array}{cc|cc}2&5&0&1\\0&-1&1&0\end{array}\right)\xrightarrow{R_1+5R_2}\left(\begin{array}{cc|cc}2&0&5&1\\0&-1&1&0\end{array}\right)\xrightarrow[R_2/-1]{R_1/2}\left(\begin{array}{cc|cc}1&0&5/2&1/2\\0&1&-1&0\end{array}\right)=(\,\mathbf{I}\mid\mathbf{F}^{-1}\,)$$

$$(\,\mathbf{G}\mid\mathbf{I}\,)=\left(\begin{array}{cc|cc}2&-7&1&0\\-1&4&0&1\end{array}\right)\xrightarrow[R_2\leftrightarrow R_1]{}\left(\begin{array}{cc|cc}-1&4&0&1\\2&-7&1&0\end{array}\right)\xrightarrow[R_2+2R_1]{}\left(\begin{array}{cc|cc}-1&4&0&1\\0&1&1&2\end{array}\right)\xrightarrow{R_1-4R_2}\left(\begin{array}{cc|cc}-1&0&-4&-7\\0&1&1&2\end{array}\right)\xrightarrow{R_1/-1}\left(\begin{array}{cc|cc}1&0&4&7\\0&1&1&2\end{array}\right)=(\,\mathbf{I}\mid\mathbf{G}^{-1}\,)$$

Matrices $\mathbf{F}$ and $\mathbf{G}$ are invertible. Their inverses are $\mathbf{F}^{-1}=1/2\begin{pmatrix}5&1\\-2&0\end{pmatrix}$ and $\mathbf{G}^{-1}=\begin{pmatrix}4&7\\1&2\end{pmatrix}$.

$$\begin{aligned}\mathbf{X}\cdot\mathbf{F}\cdot\mathbf{G}&=\mathbf{B}\\ \mathbf{X}\cdot\mathbf{F}\cdot\mathbf{G}\cdot{\color{red}\mathbf{G}^{-1}}&=\mathbf{B}\cdot{\color{red}\mathbf{G}^{-1}}\\ \mathbf{X}\cdot\mathbf{F}&=\mathbf{B}\cdot\mathbf{G}^{-1}\\ \mathbf{X}\cdot\mathbf{F}\cdot{\color{red}\mathbf{F}^{-1}}&=\mathbf{B}\cdot\mathbf{G}^{-1}\cdot{\color{red}\mathbf{F}^{-1}}\\ \mathbf{X}&=\mathbf{B}\cdot\mathbf{G}^{-1}\cdot\mathbf{F}^{-1}\end{aligned}$$

$$\mathbf{X}=\begin{pmatrix}-1&8\\-1&12\end{pmatrix}\begin{pmatrix}4&7\\1&2\end{pmatrix}1/2\begin{pmatrix}5&1\\-2&0\end{pmatrix}=1/2\begin{pmatrix}4&9\\8&17\end{pmatrix}\begin{pmatrix}5&1\\-2&0\end{pmatrix}=1/2\begin{pmatrix}2&4\\6&8\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$$

Verify that the solution is correct:

$$\mathbf{X}\cdot\mathbf{F}\cdot\mathbf{G}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0&-1\\2&5\end{pmatrix}\begin{pmatrix}2&-7\\-1&4\end{pmatrix}=\begin{pmatrix}4&9\\8&17\end{pmatrix}\begin{pmatrix}2&-7\\-1&4\end{pmatrix}=\begin{pmatrix}-1&8\\-1&12\end{pmatrix}=\mathbf{B}\ \checkmark$$

45 – Matrix algebra, matrix equations

Exercise

Solve the following matrix equations for the unknown matrix $\mathbf{X}$. The matrices $\mathbf{C}$, $\mathbf{F}$ and $\mathbf{G}$ are invertible.

a) $\mathbf{C} \cdot \mathbf{X} = \mathbf{B}$ b) $\mathbf{X} \cdot \mathbf{C} = \mathbf{B}$ c) $\mathbf{C} \cdot \mathbf{F} \cdot \mathbf{X} = \mathbf{B}$ d) $\mathbf{C} \cdot \mathbf{X} \cdot \mathbf{F} = \mathbf{B}$ e) $\mathbf{C} \cdot \mathbf{F} \cdot \mathbf{G} \cdot \mathbf{X} = \mathbf{B}$

46 – Matrix algebra, matrix equations

Example

Solve the following matrix equation for the unknown matrix $\mathbf{X}$.

$$\mathbf{F} \cdot \mathbf{X} \cdot \mathbf{G} = \mathbf{B} \qquad \mathbf{F} = \begin{pmatrix} 0 & -1 \\ 2 & 5 \end{pmatrix} \qquad \mathbf{G} = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -1 & 8 \\ -1 & 12 \end{pmatrix}$$

47 – Matrix algebra, matrix equations

Example

Solve the following matrix equation for the unknown matrix $\mathbf{X}$.

$$\mathbf{F}\cdot\mathbf{G}\cdot\mathbf{X}=\mathbf{B} \qquad \mathbf{F}=\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 5 & -1 & 2 \end{pmatrix} \qquad \mathbf{G}=\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & 0 & 2 \end{pmatrix} \qquad \mathbf{B}=\begin{pmatrix} 6 & 2 & 4 \\ 9 & 10 & 3 \\ 19 & 2 & 15 \end{pmatrix}$$

48 – Matrix algebra, matrix equations

Example

Solve the following matrix equation for the unknown matrix **X**.

a) $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ b) $\mathbf{A} \cdot \mathbf{x} = \mathbf{c}$ c) $\mathbf{A} \cdot \mathbf{x} = \mathbf{d}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 10 \\ 10 \\ 20 \end{pmatrix} \qquad \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

49 – Matrix algebra, elementary matrices

Exercise

What matrix will perform the following?

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 3 \end{pmatrix}$$

What matrix will perform the following?

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}$$

Exercise

What matrix will swap the first and the third row?

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

And what matrix will swap them back?

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} -1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix}$$

50 – Matrix algebra, elementary matrices

Exercise

What matrix will multiply the third row by five?

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ -10 & 10 & 15 \\ 4 & 2 & 1 \end{pmatrix}$$

And what matrix will change it back?

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ -10 & 10 & 15 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix}$$

Exercise

What matrix will add the first row to the third one?

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \\ 4 & 2 & 1 \end{pmatrix}$$

And what matrix inverts it back?

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix}$$

51 – Matrix algebra, elementary matrices

Definition

Elementary matrices are square matrices that can be obtained from the identity matrix by performing one single elementary row operation.

For every elementary row operation there is a elementary matrix such that multiplying by it from the left performs the operation.
For example:

- Interchanging two rows:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \cdots$$

- Multiplying a row by a nonzero number α:

$$\mathbf{E}_{11} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{E}_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{E}_{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

- Adding multiple of one row to another row.

$$\mathbf{E}_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} \quad \mathbf{E}_{12} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{E}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{pmatrix} \quad \cdots$$

Definition

A matrix that can be obtained from the identity matrix by swapping two or more rows is called **permutation matrix**.

Exercise

What row operation is performed by matrix $\mathbf{E}_{12}$?

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{E}_{12}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

52 – Matrix algebra, elementary matrices

Exercise

What are inverses of the matrices $\mathbf{E}_{12}$ and $\mathbf{E}_{23}$?

$$\mathbf{E}_{12} = \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{E}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

Exercise

Determine the matrix $\mathbf{B}$. (Start with $\mathbf{E}_{12}\mathbf{A}$.)

$$\underset{\mathbf{E}_{23}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}} \cdot \left(\underset{\mathbf{E}_{12}}{\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \cdot \underset{\mathbf{A}}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \right) = \underset{\mathbf{B}}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}$$

What elementary matrices will recover the matrix $\mathbf{A}$ from $\mathbf{B}$.

$$\underset{\mathbf{A}}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}} = \underset{?}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}} \cdot \underset{?}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}} \cdot \underset{\mathbf{B}}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}$$

53 – Matrix algebra, LU factorization

Some square matrices $\mathbf{A}$ can be decomposed into two matrices, a lower triangular matrix $\mathbf{L}$ and an upper triangular matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{L}\cdot\mathbf{U}$. Such a decomposition is called the **LU factorization** of $\mathbf{A}$. It may be found by performing Gaussian elimination. This will be demonstrated on the following example.

Example

Find the **LU** factors the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 6 & 9 \\ 3 & 5 & 13 \end{pmatrix}$$

The elimination performed on the matrix **A** produces the wanted matrix **U**.

$$\underset{\mathbf{A}}{\begin{pmatrix} 1 & 1 & 1 \\ 5 & 6 & 9 \\ 3 & 5 & 13 \end{pmatrix}} \xrightarrow{R_2-5R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 3 & 5 & 13 \end{pmatrix} \xrightarrow{R_3-3R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 2 & 10 \end{pmatrix} \xrightarrow{R_3-2R_2} \underset{\mathbf{U}}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}}$$

Each one of these row operations can be carried out as multiplication by an elementary matrix.

$$\underset{\mathbf{E}_{23}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}} \cdot \underset{\mathbf{E}_{13}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}} \cdot \underset{\mathbf{E}_{12}}{\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \cdot \underset{\mathbf{A}}{\begin{pmatrix} 1 & 1 & 1 \\ 5 & 6 & 9 \\ 3 & 5 & 13 \end{pmatrix}} = \underset{\mathbf{U}}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}}$$

$$\begin{aligned} \mathbf{E}_{23}\cdot\mathbf{E}_{13}\cdot\mathbf{E}_{12}\cdot\mathbf{A} &= \mathbf{U} \\ \mathbf{E}_{13}\cdot\mathbf{E}_{12}\cdot\mathbf{A} &= \mathbf{E}_{23}^{-1}\cdot\mathbf{U} \\ \mathbf{E}_{12}\cdot\mathbf{A} &= \mathbf{E}_{13}^{-1}\cdot\mathbf{E}_{23}^{-1}\cdot\mathbf{U} \\ \mathbf{A} &= \underbrace{\mathbf{E}_{12}^{-1}\cdot\mathbf{E}_{13}^{-1}\cdot\mathbf{E}_{23}^{-1}}_{\mathbf{L}}\cdot\mathbf{U} \end{aligned}$$

$$\mathbf{L} = \underset{\mathbf{E}_{12}^{-1}}{\begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \cdot \underset{\mathbf{E}_{13}^{-1}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}} \cdot \underset{\mathbf{E}_{23}^{-1}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

Thus

$$\underset{\mathbf{A}}{\begin{pmatrix} 1 & 1 & 1 \\ 5 & 6 & 9 \\ 3 & 5 & 13 \end{pmatrix}} = \underset{\mathbf{L}}{\begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}} \cdot \underset{\mathbf{U}}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}}$$

The matrix **L** has ones on its diagonal. Entries below the diagonal are called **multipliers**. The multiplier ℓ_{ij} is the number used in the elimination to annihilate the ij-position: $R_i - \ell_{ij}R_j$

$$\ell_{ij} = \frac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

A square matrix **A** can be decomposed into $\mathbf{L}\cdot\mathbf{U}$ if there was no need to exchange rows during the elimination. Otherwise the factorization has the form $\mathbf{P}\cdot\mathbf{A} = \mathbf{L}\cdot\mathbf{U}$, where **P** is a permutation matrix.

54 – Matrix algebra, LU factorization

LU factorization is a very useful tool for solving multiple systems with the same coefficient matrix and different right-hand sides.

Example

Use the LU factorization of $\mathbf{A}$ to solve $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 6 & 9 \\ 3 & 5 & 13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

Once the LU factors of $\mathbf{A}$ are known (see example on page 53), it is easy to solve $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} \mathbf{A}\cdot\mathbf{x} &= \mathbf{b} \\ \mathbf{L}\cdot\mathbf{U}\cdot\mathbf{x} &= \mathbf{b} \\ \mathbf{L}\cdot(\underbrace{\mathbf{U}\cdot\mathbf{x}}_{\mathbf{y}}) &= \mathbf{b} \end{aligned}$$

So insted of solving $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$, we solve two triangular systems $\mathbf{L}\cdot\mathbf{y} = \mathbf{b}$ and $\mathbf{U}\cdot\mathbf{x} = \mathbf{y}$:

First
$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}}_{\mathbf{L}} \cdot \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}}_{\mathbf{b}}$$
and then
$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}}_{\mathbf{U}} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}}_{\mathbf{y}}$$

Forward substitution to get $\mathbf{y}$:

$$\begin{aligned} y_1 &= 0 \\ y_2 &= 1 - 5y_1 = 1 \\ y_3 &= 4 - 3y_1 - 2y_2 = 2 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Backward substitution to get $\mathbf{x}$.

$$\begin{aligned} x_1 &= 0 - x_2 - x_3 = 2 \\ x_2 &= 1 - 4x_3 = -3 \\ x_3 &= 2/2 = 1 \end{aligned}$$

The solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

55 – Matrix algebra, LU factorization

Exercise

Use the LU factorization of $\mathbf{A}$ to solve $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 6 & 9 \\ 3 & 5 & 13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 27 \\ 15 \end{pmatrix}$$

56 – Matrix algebra, LU factorization

Exercise

Find the LU factors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 3 & 6 & 1 \\ -9 & -22 & 4 \\ -12 & -36 & 19 \end{pmatrix}$$

57 – Matrix algebra, LU factorization

Exercise

Find the LU factors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 6 & -3 & -9 & 3 \\ 24 & -8 & -29 & 13 \\ -18 & -11 & -10 & -9 \\ 4 & 6 & -6 & 27 \end{pmatrix}$$

58 – Matrix algebra, LU factorization

Exercise

Find the LU factors of **A** and solve systems below.

$$\begin{pmatrix} 2 & 1 & 4 \\ 4 & 1 & 15 \\ 8 & 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 15 \\ 60 \\ -15 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 4 \\ 4 & 1 & 15 \\ 8 & 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 200 \\ 400 \\ 800 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 4 \\ 4 & 1 & 15 \\ 8 & 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

59 – Matrix algebra, determinants

Definition

The **determinant** of a square 1×1 matrix $\mathbf{A} = (a_{11})$ is defined to be the number a_{11}.

The **determinant** of a square matrix $\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ is defined to be the number

$$\det(\mathbf{A}) = \sum_{k=1}^{n} a_{1k}(-1)^{1+k} M_{1k}$$

where M_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix that results from $\mathbf{A}$ by removing its ith row and its jth column.
The number $C_{ij} = (-1)^{i+j} M_{ij}$ is called the **cofactor** associated with the position ij.
The determinant of the matrix $\mathbf{A}$ is denoted $\det(\mathbf{A})$, $\det \mathbf{A}$, or $|\mathbf{A}|$.

The determinat of the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Example

Compute the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -1 & -3 \end{pmatrix}$.

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & -2 \\ -1 & -3 \end{vmatrix} = 1\cdot(-3) - (-1)\cdot(-2) = -3 - 2 = -5$$

Exercise

Compute the determinant of the matrix $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 3 & 6 \end{pmatrix}$.

Exercise

Compute the determinant of the matrix $\mathbf{C} = \begin{pmatrix} 1 & 15 \\ 2 & 1 \end{pmatrix}$.

60 – Matrix algebra, determinants

Example

Compute the determinant of the matrix $\begin{pmatrix} 2 & 1 & 3 \\ 5 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$.

$$\det(\mathbf{A}) = 2\cdot(-1)^2 \cdot \begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} + 1\cdot(-1)^3 \cdot \begin{vmatrix} 5 & 0 \\ 1 & 3 \end{vmatrix} + 3\cdot(-1)^4 \cdot \begin{vmatrix} 5 & 0 \\ 1 & 2 \end{vmatrix} =$$

$$= 2\cdot 0 + 1\cdot(-15) + 3\cdot 10 = 15$$

Rule of Sarrus for the determinant of a 3×3 matrix $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\det(\mathbf{A}) = \left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right| = aei + bfg + cdh - ceg - afh - bdi$$

Example

Compute the determinant of the matrix $\begin{pmatrix} 2 & 1 & 3 \\ 5 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$.

$$\det(\mathbf{A}) = \left|\begin{array}{ccc|cc} 2 & 1 & 3 & 2 & 1 \\ 5 & 0 & 0 & 5 & 0 \\ 1 & 2 & 3 & 1 & 2 \end{array}\right| =$$

$$= 2\cdot 0\cdot 3 + 1\cdot 0\cdot 1 + 3\cdot 5\cdot 2 - 1\cdot 0\cdot 3 - 2\cdot 0\cdot 2 - 3\cdot 5\cdot 1 =$$

$$= 0 + 0 + 30 - 0 - 0 - 15 = 15$$

Exercise

Compute the determinant of the matrix $\begin{pmatrix} 1 & 5 & 2 \\ 0 & 6 & 3 \\ -8 & 2 & 5 \end{pmatrix}$.

Exercise

Compute the determinant of the matrix $\begin{pmatrix} 4 & 2 & 1 \\ 0 & 0 & 3 \\ 2 & 0 & 1 \end{pmatrix}$.

61 – Matrix algebra, determinants

The determinant of the matrix $\mathbf{A}_{n\times n}$ can be expressed by the folowing so called **Laplace expansions** or **cofactor expansions**.

$$\det(\mathbf{A}) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad \text{expansion along the } i\text{th row}$$
$$\det(\mathbf{A}) = a_{1j}C_{1j} + a_{nj}C_{2j} + \cdots + a_{nj}C_{nj} \qquad \text{expansion along the } j\text{th column}$$

Example

Find the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$ by expanding it

a) along the third column

b) along the second row

a) $$\det(\mathbf{A}) = 3\cdot(-1)^4 \cdot \begin{vmatrix} 5 & 0 \\ 1 & 2 \end{vmatrix} + 0\cdot(-1)^5 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 3\cdot(-1)^6 \cdot \begin{vmatrix} 2 & 1 \\ 5 & 0 \end{vmatrix} =$$

$$= 3\cdot 10 - 0 + 3\cdot(-5) = 15$$

b) $$\det(\mathbf{A}) = 5\cdot(-1)^3 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} + 0\cdot(-1)^4 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} + 0\cdot(-1)^5 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} =$$

$$= 5\cdot(-1)\cdot(-3) + 0 + 0 = 15$$

62 – Matrix algebra, determinants

Exercise

Use Laplace expansion to evaluate the determinant of the following matrices.

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 0 & 3 \\ 2 & 0 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 5 & 2 \\ 0 & 6 & 3 \\ -8 & 2 & 5 \end{pmatrix}$$

63 – Matrix algebra, determinants

Example

Compute the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & 3 \\ 4 & 0 & 1 & 0 \\ -1 & 0 & 2 & 3 \\ 1 & -2 & 1 & 1 \end{pmatrix}$.

Expansion along the second row: $\det(\mathbf{A}) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24}$

$$\det(\mathbf{A}) = 4\cdot(-1)^3\cdot\begin{vmatrix} -2 & 1 & 3 \\ 0 & 2 & 3 \\ -2 & 1 & 1 \end{vmatrix} + 0\cdot(-1)^4\cdot\begin{vmatrix} 1 & 1 & 3 \\ -1 & 2 & 3 \\ -1 & 1 & 1 \end{vmatrix} + 1\cdot(-1)^5\cdot\begin{vmatrix} 1 & -2 & 3 \\ -1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} + 0\cdot(-1)^6\cdot\begin{vmatrix} 1 & -2 & 1 \\ -1 & 0 & 2 \\ 1 & -2 & 1 \end{vmatrix} =$$

$$= 4\cdot(-1)\cdot(-4-6+0+12+6-0) + 0 + 1\cdot(-1)\cdot(0-6+6-0+6-2) + 0 =$$

$$= (-4)\cdot(8) + (-1)\cdot(4) = -32 - 4 = -36$$

64 – Matrix algebra, determinants

Exercise

Compute the determinant of the following matrices.

$$\mathbf{A} = \begin{pmatrix} 0 & 3 & 1 & 0 \\ -1 & 0 & 2 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & -2 & -1 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & -1 & 4 & 0 \end{pmatrix}$$

Worksheets for Mathematics I

Vector spaces $\mathbb{R}^n$

Jana Bělohlávková

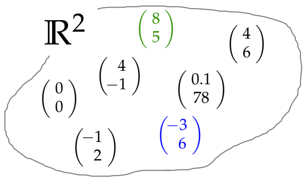
66 – Vector spaces $\mathbb{R}^n$

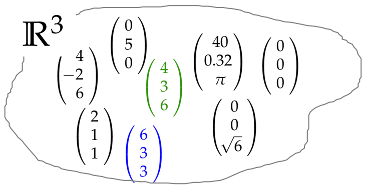
Definition

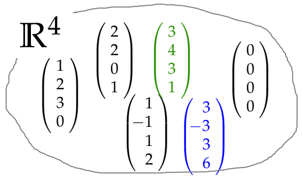
The **vector space** $\mathbb{R}^n$ consist of all vectors $\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ with n real entries.

Any two vectors can be added together and any vector can be multiplied by a real number.

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix} \qquad c \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} c \cdot u_1 \\ c \cdot u_2 \\ \vdots \\ c \cdot u_n \end{pmatrix} \qquad \textbf{zero vector} \quad \mathbf{o} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

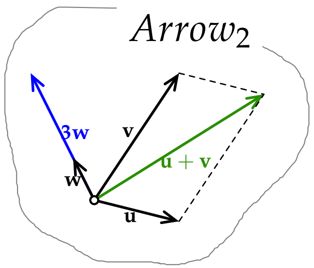


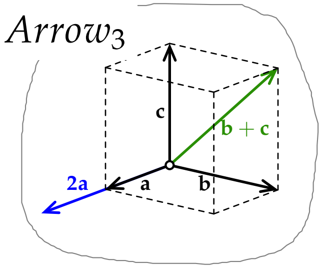


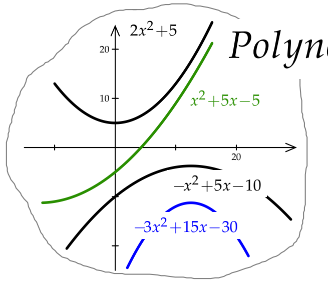


$\mathbb{R}^5, \mathbb{R}^6, \ldots$

Although we will only study the vector spaces $\mathbb{R}^n$, there are other sets of objects that also form vector spaces.





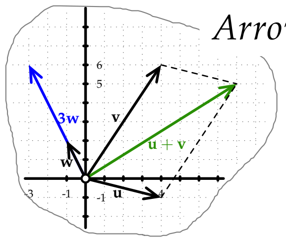


$Polynom_2$

$Polynom_3, Polynom_4, \ldots$

All their objects (vectors) have some things in common. They can be added together, multiplied by a number, there is a "zero" vector among them, etc.

There is a natural correspondence between some vector spaces. For example $Arrows_2 \sim \mathbb{R}^2$, $Arrows_3 \sim \mathbb{R}^3$. There is no corresponding "arrow" space for $\mathbb{R}^4, \mathbb{R}^5, \ldots$



$Arrows_2 \sim \mathbb{R}^2$

67 – Vector spaces $\mathbb{R}^n$, linear combination and span

Definition

The vector $\mathbf{w}$ is a **linear combination** of the vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_r}$ if there are numbers $c_1, c_2, \dots c_r$ such that
$$\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_r\mathbf{v_r}.$$

The vector $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ is a linear combination of the vectors $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ since

$$\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

This can be writtten as matrix equation:

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$$

Definition

For a set of vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_r}\}$ the **span of** S is the set of all linear combinations of vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_r}$. It is denoted $\operatorname{span}(S)$.

68 – Vector spaces $\mathbb{R}^n$, linear combination and span

Example

Is the vector $\mathbf{w}$ a linear combination of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$?

$$\mathbf{w} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v_1} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 1 \end{pmatrix}.$$

We are looking for three numbers c_1, c_2, c_3 such that

$$c_1 \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + c_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

This can be written as the matrix equation

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

and solved by elimination.

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_4} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

There is no solution, therefore the vector $\mathbf{w}$ is not a linear combination of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$. In other words, the vector $\mathbf{w}$ is not in the span of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

Example

Is the vector $\mathbf{w}$ a linear combination of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$?

$$\mathbf{w} = \begin{pmatrix} 7 \\ 0 \\ 7 \\ 2 \end{pmatrix}, \quad \mathbf{v_1} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 1 \end{pmatrix}.$$

We are looking for three numbers c_1, c_2, c_3 such that

$$c_1 \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + c_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 7 \\ 2 \end{pmatrix}$$

This can be written as the matrix equation

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 7 \\ 2 \end{pmatrix}$$

and solved by elimination.

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 7 \\ 0 & 1 & 1 & 2 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_4} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 5 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Back substitution gives the solution $c_1 = 3$, $c_2 = 1$, $c_3 = 1$. Therefore the vector $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$.

$$\mathbf{w} = 3\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}.$$

69 – Vector spaces $\mathbb{R}^n$, linear independence

Definition

The sequence of vectors $\langle \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r} \rangle$ is called **linearly independent** if the only solution to the equation

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_r\mathbf{v_r} = \mathbf{o}$$

is $c_1 = c_2 = \cdots = c_r = 0$.
If there is a solution with at least one nonzero c_i, the sequence of vectors is called **linearly dependent**.

Example

Decide whether the sequence of vectors $\langle \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5} \rangle$ is linearly independent.

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 4 \\ 0 \\ 8 \\ 4 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v_4} = \begin{pmatrix} 3 \\ 4 \\ 8 \\ 3 \end{pmatrix}, \quad \mathbf{v_5} = \begin{pmatrix} 4 \\ 2 \\ 9 \\ 4 \end{pmatrix}.$$

$$\left(\begin{array}{ccccc|c} 1 & 4 & 1 & 3 & 4 & 0 \\ 0 & 0 & 2 & 4 & 2 & 0 \\ 2 & 8 & 3 & 8 & 9 & 0 \\ 1 & 4 & 1 & 3 & 4 & 0 \end{array}\right) \begin{array}{l} \\ \\ R_3-2R_1 \\ R_4-R_1 \end{array} \to \left(\begin{array}{ccccc|c} 1 & 4 & 1 & 3 & 4 & 0 \\ 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ R_2/2 \\ 2R_3-R_2 \\ \\ \end{array} \to \left(\begin{array}{ccccc|c} 1 & 4 & 1 & 2 & 4 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1-R_2 \\ \\ \\ \\ \end{array} \to \left(\begin{array}{ccccc|c} 1 & 4 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The system has infinitely many solutions.

$$-4\mathbf{v_1} + \mathbf{v_2} + 0\mathbf{v_3} + 0\mathbf{v_4} + 0\mathbf{v_5} = \mathbf{o}, \qquad 1\mathbf{v_1} + 0\mathbf{v_2} + 2\mathbf{v_3} - \mathbf{v_4} + 0\mathbf{v_5} = \mathbf{o}, \qquad \ldots$$

Therefore the sequence $\langle \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5} \rangle$ is lineary dependent.

From the reduced row echelon form we can see that vectors $\mathbf{v_1}$ and $\mathbf{v_3}$ are in the basic columns. They are lineary independent. The other vectors are their linear combination.

$$\begin{aligned} \mathbf{v_2} &= 4\mathbf{v_1} \\ \mathbf{v_4} &= \mathbf{v_1} + 2\mathbf{v_3} \\ \mathbf{v_5} &= 3\mathbf{v_1} + \mathbf{v_3} \end{aligned}$$

70 – Vector spaces $\mathbb{R}^n$, linear independence

Exercise

Decide whether the sequence of vectors $\langle \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5} \rangle$ is linearly independent.

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 6 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v_4} = \begin{pmatrix} 5 \\ 1 \\ 6 \\ 11 \end{pmatrix}, \quad \mathbf{v_5} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 6 \end{pmatrix}.$$

71 – Vector spaces $\mathbb{R}^n$, metric structure

Definition

For two vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ of $\mathbb{R}^n$ their **dot product** (or the **standard inner product**) is defined to be the number

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example

Find the dot product of the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}$.

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 0 + 2 \cdot (-5) + 3 \cdot 1 = 0 - 10 + 3 = -7$$

Exercise

Find the dot product of the vectors $\mathbf{u} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$.

The dot product able us to "measure" the magnitude and the angle of vectors of $\mathbb{R}^n$.

Definition

For the vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ of $\mathbb{R}^n$ the **magnitude of vector** (or the **euclidean vector norm**) is defined to be a number

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Example

Find the magnitude of the vector $\mathbf{u} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$.

$$\|\mathbf{u}\| = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26} \doteq 5.09$$

Exercise

Find the magnitude of the vector $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$.

72 – Vector spaces $\mathbb{R}^n$, metric structure

Definition

For two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ of $\mathbb{R}^n$ the **angle** between them is defined to be the number $\varphi \in \langle 0, \pi \rangle$ such that

$$\cos\varphi = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Example

Find the angle of the vectors $\mathbf{u} = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

$$\mathbf{u}\cdot\mathbf{v} = 5\cdot 2 + (-4)\cdot 1 + 3\cdot 3 = 15$$

$$\|\mathbf{u}\| = \sqrt{5^2 + (-4)^2 + 3^2} = \sqrt{50}$$

$$\|\mathbf{v}\| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

Enter the results into the formula.

$$\cos\varphi = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{15}{\sqrt{50}\sqrt{14}} \doteq 0.5669$$

$$\varphi = 0.96 \text{ rad} \qquad \varphi = 55.46^\circ$$

Exercise

Find the angle of vectors $\mathbf{u}$ and $\mathbf{v}$.

a) $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$

b) $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$

73 – Vector spaces $\mathbb{R}^n$, metric structure

Definition

Two vectors of $\mathbb{R}^n$ are said to be **orthogonal** or **perpendicular** (to each other) whenever their dot product equals zero.

$$\mathbf{u} \cdot \mathbf{v} = 0$$

Exercise

Deside, whether the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

a) $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$

b) $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$

Exercise

Fill missing numbers, so that the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

a) $\mathbf{u} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -5 \\ 7 \\ * \end{pmatrix}$

b) $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} * \\ 2 \\ 1 \end{pmatrix}$

Worksheets for Mathematics I

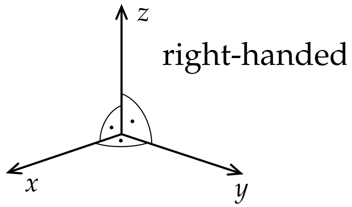
Euclidean space $\mathbb{E}_3$

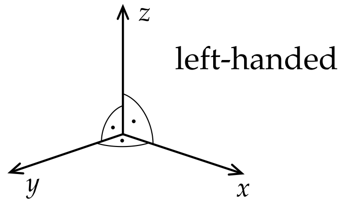
Jana Bělohlávková

75 – Euclidean space $\mathbb{E}_3$

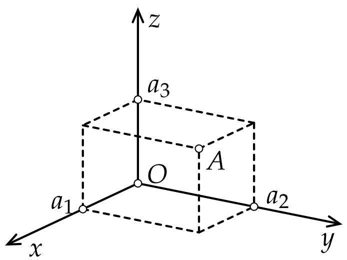
Definition

Cartesian coordinate system (in 3D) consists of an ordered triplet of orientated lines (**the axes**) pair-wise perpendicular that go through a common point (**the origin**) and are pair-wise perpendicular; and a single unit of length common for all three axes.
The axes are denoted x, y, z.
xy-plane, yz-plane, xz-plane are called **coordinate planes**.
System can be either **right-handed** or **left-handed**





In 3D space equipped with a Cartesian coordinate system, every point A is uniquely determined by an ordered triplet of numbers $[a_1, a_2, a_3]$ as shown in the picture below. The numbers are called **coordinates** of the point A. We write this as $A = [a_1, a_2, a_3]$.



From now on to conserve space we will write vectors from the space $\mathbb{R}^3$ differently. Instead of $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ we will write $\mathbf{u} = (u_1, u_2, u_3)$ (numbers "laying down" and divided by commas).

Definition

Euclidean space $\mathbb{E}_3$ contains two sets of object. The set of all points $[a_1, a_2, a_3]$ and the set of all vectors (u_1, u_2, u_3) from the vector space $\mathbb{R}^3$ equipped with the dot product.
Both sets are "tied up" together, we can "add" point to a vector to get another point

$$A + \mathbf{u} = [a_1 + u_1, a_2 + u_2, a_3 + u_3].$$

For every (ordered) pair of points $A = [a_1, a_2, a_3]$ and $B = [b_1, b_2, b_3]$ there is a unique vector

$$\mathbf{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3).$$

The distance of two points A and B is the magnitude of the vector $\mathbf{AB}$.

Example

For the given points $A = [2, -1, 5]$, $B = [4, 1, 0]$ find the vector $\mathbf{AB}$.

$$\mathbf{AB} = (4-2, 1-(-1), 0-5) = (2, 2, -5)$$

Example

For the given point $A = [3, 4, 1]$ and given vector $\mathbf{u} = (2, -2, 1)$ find the coordinates of the following points:

a) $A + \mathbf{u}$, b) $A + 2\mathbf{u}$, c) $A + 3\mathbf{u}$, d) $A - \mathbf{u}$.

76 – Euclidean space $\mathbb{E}_3$

Recall that two vectors are perpendicular if their dot product is zero.
For example: Vectors perpendicular to the vector $(1,0,2)$ are $(-2,0,1)$, $(4,5,-2)$ ...,
vectors perpendicular to the vector $(3,4,1)$ are e $(1,-1,1)$, $(0,1,-4)$ ....
Is there a way to find a vector that will be perpendicular to both of them?

Definition

For two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ of $\mathbb{R}^3$ their **cross product** is defined to be the vector

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$$

The cross product $\mathbf{u} \times \mathbf{v}$ is perpendicular to the vector $\mathbf{u}$ and to the vector $\mathbf{v}$.

The cross product can also be expressed as the formal determinant

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i} \cdot \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j} \cdot \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k} \cdot \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

where $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$.

Example

Find the cross product of the vectors $\mathbf{u} = (1,0,2)$ and $\mathbf{v} = (3,4,1)$. Check that the cross product is perpendicular to the vector $\mathbf{u}$ and to the vector $\mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 3 & 4 & 1 \end{vmatrix} = \left(\begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} \right) = (-8, 5, 4)$$

To check whether they are perpendicular we calculate their dot product:
$(1,0,2)\cdot(-8,5,4) = -8+0+8 = 0$ Vectors $\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$ are perpendicular.
$(3,4,1)\cdot(-8,5,4) = -24+20+4 = 0$ Vectors $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ are perpendicular.

Exercise

Find the cross product of the vectors $\mathbf{u} = (4,-1,2)$ and $\mathbf{v} = (2,0,5)$. Check that the cross product is perpendicular to the vector $\mathbf{u}$ and to the vector $\mathbf{v}$.

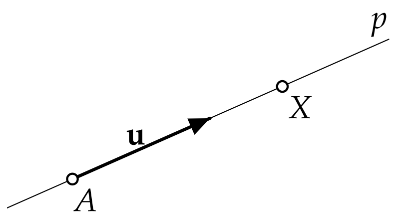
77 – Euclidean space $\mathbb{E}_3$, line

Definition

A **line** through a point $A=[a_1,a_2,a_3]$ in direction of a vector $\mathbf{u}=(u_1,u_2,u_3)$ is defined to be a set of all points X satisfying equation

$$X = A + t\mathbf{u},$$

where t is any real number.



Equations

$$\begin{aligned} x &= a_1 + tu_1 \\ p: y &= a_2 + tu_2 \\ z &= a_3 + tu_3, \quad t \in \mathbb{R}. \end{aligned}$$

are called **parametric equations of the line**.

Example

Set parametric equations for a line which passes through the point $A=[3,4,1]$ in direction of the vector $\mathbf{u}=(2,-2,1)$.

$$\begin{aligned} x &= 3 + 2t \\ p: y &= 4 - 2t \\ z &= 1 + t, \quad t \in \mathbb{R} \end{aligned}$$

Exercise

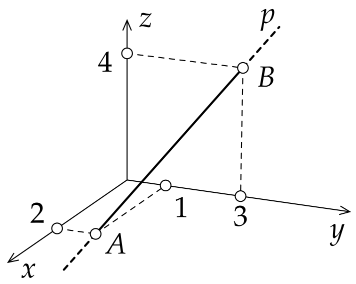
a) Find the parametric equations for a line p which passes through the point $A=[1,4,-1]$ in direction of the vector $\mathbf{u}=(3,0,2)$.

b) Find the coordinates of any four points the line p is passing through.

c) Find the missing coordinates of points $[-2,*,*]$, $[*,*,7]$ and $[*,6,*]$ so they lie on the line p.

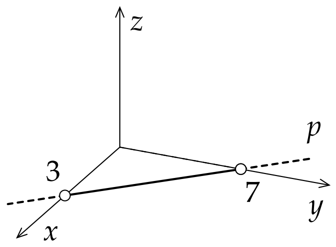
78 – Euclidean space $\mathbb{E}_3$, line

Exercise



Find the parametric equations for a line p passing through points A and B.

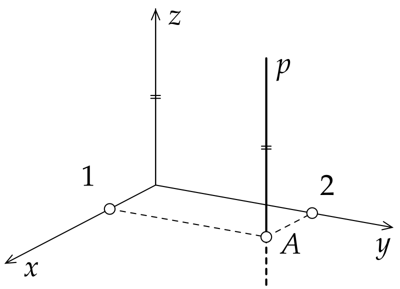
Exercise



Find the parametric equations for a line p passing through given points.

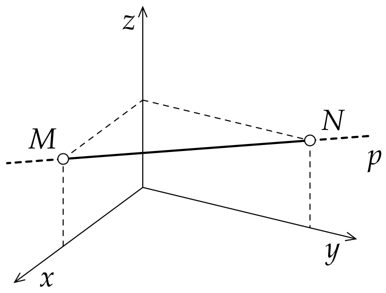
79 – Euclidean space $\mathbb{E}_3$, line

Exercise



Find the parametric equations for the line p passing through the point A and parallel with z-axis.

Exercise



Find the coordinates of the point M and the point N lying on the line p and also in xz-plane and yz-plane, respectively.

$$p: \begin{aligned} x &= 8 - 2t \\ y &= -9 + 3t \\ z &= 5 \end{aligned}$$

80 – Euclidean space $\mathbb{E}_3$, line

Exercise

Find out whether the line p passes through the point A, B or C.

$$p: \begin{aligned} x &= -1+3r \\ y &= 2-2r \\ z &= 1+r \end{aligned} \qquad \begin{aligned} A &= [2,0,2] \\ B &= [5,1,3] \\ C &= [-7,6,-1] \end{aligned}$$

Exercise

Decide whether there is a line which passes through all tree points $A=[1,1,1]$, $B=[4,3,5]$ and $C=[7,5,9]$.
If so, write its parametric equations.

81 – Euclidean space $\mathbb{E}_3$, line – line intersection

Definition

If there is a one point lying on two lines it is called their **intersection** or a (**point of intersection**).

Example

Find the intersection of lines p and q (if there is any).

$$p: \begin{aligned} x &= 8-3t \\ y &= 1-t \\ z &= 3+2t \end{aligned} \qquad\qquad q: \begin{aligned} x &= 3+s \\ y &= s \\ z &= 8+s \end{aligned}$$

Denote intersection point $P=[p_1,p_2,p_3]$. Since P lies on both lines, its coordinates must satysfied both equations for some t and some s.

$$\begin{aligned} p_1 &= 8-3t \\ p_2 &= 1-t \\ p_3 &= 3+2t \end{aligned} \qquad\qquad \begin{aligned} p_1 &= 3+s \\ p_2 &= s \\ p_3 &= 8+s \end{aligned} \qquad\qquad \begin{aligned} 8-3t &= 3+s \\ 1-t &= s \\ 3+2t &= 8+s \end{aligned}$$

We get three linear equations in two unknowns.

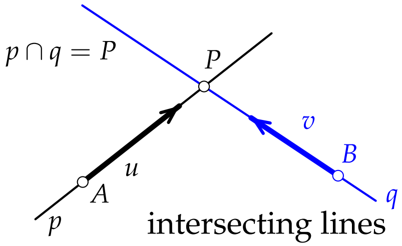
$$\begin{aligned} -3t-s &= -5 \\ -t-s &= -1 \\ 2t-s &= 5 \end{aligned} \qquad\qquad \left(\begin{array}{rr|r} -3 & -1 & -5 \\ 0 & 2 & -2 \\ 0 & -5 & 5 \end{array}\right) \to \left(\begin{array}{rr|r} -3 & -1 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

System has a one solution $t=2$ a $s=-1$. Lines intersect at one point.

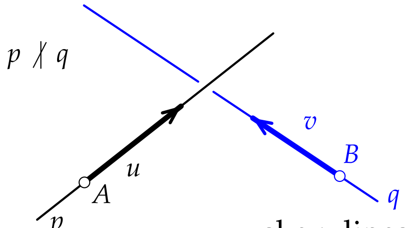
$$\begin{aligned} p_1 &= 8-3t = 8-3\cdot 2 = 2 \\ p_2 &= 1-t = 1-2 = -1 \\ p_3 &= 3+2t = 3+2\cdot 2 = 7 \end{aligned} \qquad\qquad \begin{aligned} p_1 &= 3+s = 3+(-1) = 2 \\ p_2 &= s = -1 \\ p_3 &= 8+s = 8-1 = 7 \end{aligned}$$

Lines p and q do intersect. Their point of intersection is $P=[2,-1,7]$.

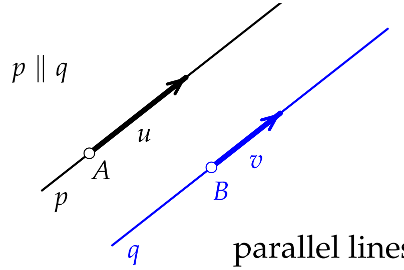
82 – Euclidean space $\mathbb{E}_3$, line – line intersection



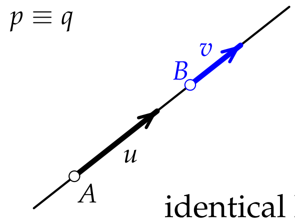
intersecting lines



skew lines



parallel lines



identical lines

Example

Decide whether lines p, q are identical, parallel, skewed or intersecting.

$$p: \begin{aligned} x &= 2+t \\ y &= 3+t \\ z &= 5+2t \end{aligned} \qquad q: \begin{aligned} x &= -3+r \\ y &= 6-r \\ z &= 7-r \end{aligned}$$

$$\begin{aligned} 2+t &= -3+r \\ 3+t &= 6-r \\ 5+2t &= 7-r \end{aligned} \qquad \begin{aligned} t-r &= -5 \\ t+r &= 3 \\ 2t+r &= 2 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & -1 & -5 \\ 1 & 1 & 3 \\ 2 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

The system has one solution $r=4$, $t=-1$.
The **lines intersect** in one point $[1,2,3]$.

Example

Decide whether lines p, q are identical, parallel, skewed or intersecting.

$$p: \begin{aligned} x &= 1+t \\ y &= 1-2t \\ z &= -2+3t \end{aligned} \qquad q: \begin{aligned} x &= 2-r \\ y &= 2+3r \\ z &= -5-4r \end{aligned}$$

$$\begin{aligned} 1+t &= 2-r \\ 1-2t &= 2+3r \\ -2+3t &= -5-4r \end{aligned} \qquad \begin{aligned} t+r &= 1 \\ -2t-3r &= 1 \\ 3t+4r &= -3 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & -3 & 1 \\ 3 & 4 & -3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -9 \end{array}\right)$$

The system has no solution. The lines don't intersect. Therefore they are eather parallel or skew. Since their direction vectors $(1,-2,3)$ and $(-1,3,-4)$ don't have same direction the **lines are skewed**.

Example

Decide whether lines p, q are identical, parallel, skewed or intersecting.

$$p: \begin{aligned} x &= 1+t \\ y &= 1-2t \\ z &= -2+3t \end{aligned} \qquad q: \begin{aligned} x &= 2+2r \\ y &= 2-4r \\ z &= -5+6r \end{aligned}$$

$$\begin{aligned} 1+t &= 2+2r \\ 1-2t &= 2-4r \\ -2+3t &= -5+6r \end{aligned} \qquad \begin{aligned} t-2r &= 1 \\ -2t+4r &= 1 \\ 3t-6r &= -3 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & -2 & 1 \\ -2 & 4 & 1 \\ 3 & -6 & -3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

The system has no solution. The lines don't intersect. Therefore they are eather parallel or skew. Since their direction vectors $(1,-2,3)$ and $(2,-4,6)$ have the same direction the **lines are parallel**.

Example

Decide whether lines p, q are identical, parallel, skewed or intersecting.

$$p: \begin{aligned} x &= 1+t \\ y &= 1-2t \\ z &= -2+3t \end{aligned} \qquad q: \begin{aligned} x &= 2-3r \\ y &= -1+6r \\ z &= 1-9r \end{aligned}$$

$$\begin{aligned} 1+t &= 2-3r \\ 1-2t &= -1+6r \\ -2+3t &= 1-9r \end{aligned} \qquad \begin{aligned} t+3r &= 1 \\ -2t-6r &= -2 \\ 3t+9r &= 3 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & 3 & 1 \\ -2 & -6 & -2 \\ 3 & 9 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The system has infitely meny solutions. The lines have infinitely many common points. Therefore they are **identical**.

83 – Euclidean space $\mathbb{E}_3$, line – line intersection

Exercise

Decide whether lines p, q are identical, parallel, skewed or intersecting.

$$p: \begin{aligned} x &= 1 + 2t \\ y &= 7 - 6t \\ z &= -2 + 8t \end{aligned} \qquad q: \begin{aligned} x &= -2 - s \\ y &= 10 + 3s \\ z &= 1 - 4s \end{aligned}$$

Exercise

Decide whether lines p, q are identical, parallel, skewed or intersecting.

$$p: \begin{aligned} x &= 7 + t \\ y &= -11 + 3t \\ z &= -10 + 3t \end{aligned} \qquad q: \begin{aligned} x &= 5 + 6s \\ y &= 3 - 2s \\ z &= 10 + s \end{aligned}$$

84 – Euclidean space $\mathbb{E}_3$, line – line intersection

Exercise

Decide whether lines p, q are identical, parallel, skewed or intersecting.

$$p: \begin{aligned} x &= 1 + t \\ y &= 1 - 2t \\ z &= 2 + 2t \end{aligned} \qquad q: \begin{aligned} x &= 2 - s \\ y &= -1 + s \\ z &= 1 - s \end{aligned}$$

Exercise

Decide whether lines p, q are identical, parallel, skewed or intersecting.

$$p: \begin{aligned} x &= 10 + t \\ y &= -7 - t \\ z &= 3 + 3t \end{aligned} \qquad q: \begin{aligned} x &= 14 - 2s \\ y &= 3 + 2s \\ z &= 15 - 6s \end{aligned}$$

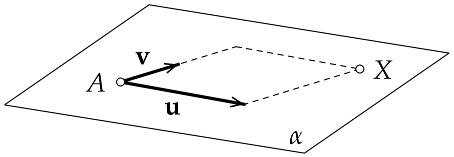
85 – Euclidean space $\mathbb{E}_3$, plane

Definition

A **plane** passing through the point $A = [a_1, a_2, a_3]$ in direction of two independent vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined to be the set of all points X such that

$$X = A + t\mathbf{u} + s\mathbf{v}$$

for some real numbers t, s.



Vectors **u** and **v** are called **direction vectors** of the plane α.

The equations
$$\begin{aligned} & x = a_1 + tu_1 + sv_1 \\ \alpha : \; & y = a_2 + tu_2 + sv_2 \\ & z = a_3 + tu_3 + sv_3 \qquad t, s \in \mathbb{R}. \end{aligned}$$

are called the **parametric equations of the plane**.
A vector is said to **lie in the plane** if it is a linear combination of **u** and **v**.

Exercise

Find coordinates of the points B, C and D;
$A = [2, 3, 1]$, $\mathbf{u} = (0, 1, 3)$, $\mathbf{v} = (1, 2, 1)$.

a) $B = A + \mathbf{u} + \mathbf{v}$, b) $C = A + 2\mathbf{u} + \mathbf{v}$, c) $D = A + \mathbf{u} + 3\mathbf{v}$.

Example

Find parametric equations for the plane which passes through the point $[7, -2, 3]$ in direction of vectors $(1, 5, 3)$ and $(-4, 1, -2)$.

$$\begin{aligned} x &= 7 + t - 4s \\ y &= -2 + 5t + s \\ z &= 3 + 3t - 2s \qquad t, s \in \mathbb{R}. \end{aligned}$$

Example

Find parametric equations for the plane which passes through the points $[1, 1, 1]$, $[4, 3, 5]$, $[7, 5, 9]$.

86 – Euclidean space $\mathbb{E}_3$, plane

Exercise

a) Find parametric equations for a plane α which passes through the point $A=[1,2,-2]$ in direction of $\mathbf{u}=(3,-1,-1)$ a $\mathbf{v}=(1,0,-2)$.

b) Find the coordinates of any four random points lying in the plane α.

c) Find the missing coordinates of points $[*,0,0]$, $[0,0,*]$ of the plane α.

Example

Convert parametric equations of the plane α (from previous exercise) into one equation by eliminating t and s.

$$\begin{aligned} x &= 1+3t+s \\ \alpha : y &= 2-t \\ z &= -2-t-2s, \quad t,s\in\mathbb{R} \end{aligned}$$

Myltiply first equation by two, second by five and add all equations together.

$$\begin{aligned} 2x &= 2+6t+2s \\ 5y &= 10-5t \\ z &= -2-t-2s \\ \hline 2x+5y+z &= 10+0t+0s \end{aligned}$$

The result is $$2x+5y+z-10=0$$

Definition

Equation $$ax+by+cz+d=0$$

is called the **general form** of the equation of the plane.
Numbers a, b, c must not be all zero.

Example

Check whether point $A=[1,2,-2]$ lies in $\alpha:\ 2x+5y+z-10=0$.

$2\cdot 1+5\cdot 2+1\cdot(-2)-10=0$. Yes, it does.

Exercise

Find the missing coordinates of points $[*,\ 0,\ 0]$, $[0,\ 0,\ *]$ of the plane $\alpha: 2x+5y+z-10=0$.

87 – Euclidean space $\mathbb{E}_3$, plane

Conclusion form previous:
Plane α passes through the point A in direction of vectors $\mathbf{u}$ a $\mathbf{v}$.
$A=[1,\ 2,-2]$, $\mathbf{u}=(3,-1,-1)$, $\mathbf{v}=(1,\ 0,-2)$.
has general equation $\alpha : 2x+5y+1z-10=0$.

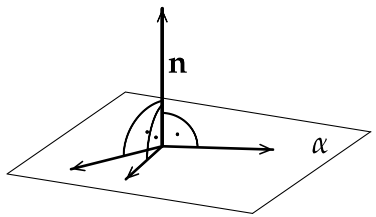
Notice that the vector $\mathbf{n}=(2,5,1)$ is perpendicular to vectors $\mathbf{u}$ and $\mathbf{v}$.

Definition

A vector $\mathbf{n}$ is said to be **perpendicular to a plane** if it is perpendicular to all vectors that lie in the plane. Any such vector is called a **normal vector** of the plane.

A normal vector of the plane $\alpha :\ \ ax+by+cz+d=0$ is the vector (a,b,c).

A normal vector of the plane $\alpha :\ 2x+5y+z-10=0$ is the vector $\mathbf{n}=(2,5,1)$ and also any scalar multiple of $\mathbf{n}$: $(4,10,2)$, $(-2,-5,-1)$,...



$$\alpha :\quad ax+by+cz+d=0$$
$$\mathbf{n}=(a,b,c)$$

Exercise

Find a general equation of a plane with a normal vector $(-3,1,4)$ and passing through point $[1,2,1]$.

Example

Find the general equation of a plane that goes through the point $A=[1,2,-2]$ in direction of vectors $\mathbf{u}=(3,-1,-1)$ and $\mathbf{v}=(1,0,-2)$.

The equation we are looking for is in the form $ax+by+cz+d=0$, where (a,b,c) is a normal vector. A normal vector is one that is perpendicular to $\mathbf{u}$ and $\mathbf{v}$, so their vector product $\mathbf{u}\times\mathbf{v}$ is a normal vector.

$$(a,b,c)=\mathbf{u}\times\mathbf{v}=\left(\begin{vmatrix}-1 & -1\\ 0 & -2\end{vmatrix}, -\begin{vmatrix}3 & -1\\ 1 & -2\end{vmatrix}, \begin{vmatrix}3 & -1\\ 1 & 0\end{vmatrix}\right)=(2,5,1)$$

We know that $a=2$, $b=5$ and $c=1$. To find the number d we use the fact that the point A lies in the plane. Therefore

$$\begin{aligned}2\cdot(1)+5\cdot(2)+(-2)+d&=0\\ d&=-10\end{aligned}$$

A general equation of the plane is

$$2x+5y+z-10=0$$

88 – Euclidean space $\mathbb{E}_3$, plane

Exercise

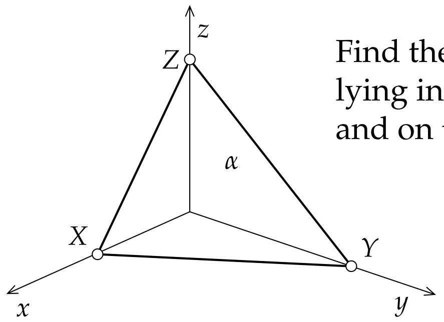
Find the general equation of the plane that goes through the point $A = [-2, 0, 5]$ in direction of vectors $\mathbf{u} = (4, 2, -1)$ a $\mathbf{v} = (-1, 1, 2)$.
Check whether points $D = [5, 5, 5]$ and $E = [6, 6, 6]$ lie in the plane.

Exercise

Find the general equation of the plane that goes through the points $A = [3, 1, 5]$, $B = [4, 2, 7]$ and $C = [5, 3, 9]$.
Find the coordinates of any four random points lying in the plane.

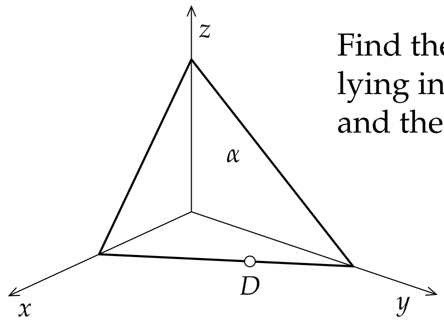
89 – Euclidean space $\mathbb{E}_3$, plane

Exercise



Find the coordinates of points X, Y and Z lying in the plane $\alpha: \; 4x + 6y + z - 12 = 0$ and on the axes.

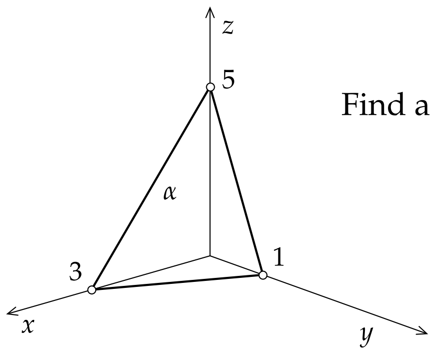
Exercise



Find the coordinates of any point D lying in the plane $\alpha: \; 4x + 6y + z - 12 = 0$ and the xy-plane.

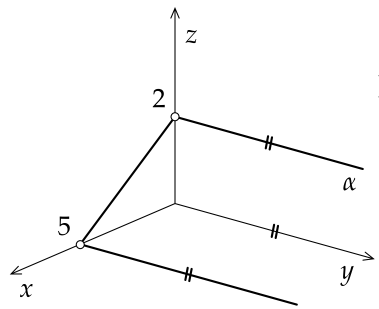
90 – Euclidean space $\mathbb{E}_3$, plane

Exercise



Find a general equation of the plane α.

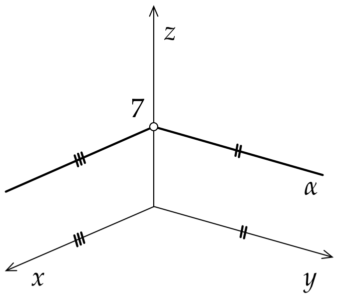
Exercise



Find a general equation of the plane α.

91 – Euclidean space $\mathbb{E}_3$, plane

Exercise



Find a general equation of the plane α parallel with the xy-plane.

Exercise

Find a general equation of the plane α perpendicular to the x-axis and passing through the point $Q=[1,-2,3]$.

92 – Euclidean space $\mathbb{E}_3$, plane

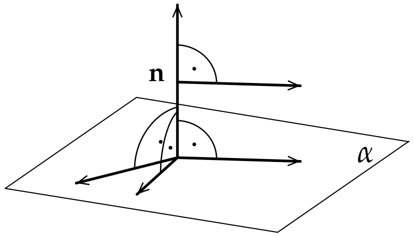
Exercise

Do points $[1,1,1]$, $[1,0,4]$, $[2,1,0]$ lie in the plane α?
Do they lie in the plane β?

$$\alpha : x + 3y + z - 5 = 0 \qquad\qquad \beta : 2x + 6y + 2z - 10 = 0$$

Example

Do vectors $(1,5,9)$, $(6,4,3)$ lie in the plane $\alpha :\ x - 3y + 2z + 2 = 0$?
Write few other vectors which do lie in the plane.



Every vector lying in a plane is perpendicular to a normal vector of the plane. This can be tested by their dot product.

$(1,5,9) \cdot (1,-3,2) \neq 0$ vector (1,5,9) is not in the plane α

$(6,4,3) \cdot (1,-3,2) = 0$ vector (6,4,3) lies in the plane α

Instead "vector lies in a plane" one can say "vector is parallel with a plane". It is the same.

Example

Find the line of intersection between the two planes α, β (if there is any).

$$\alpha : x - y + 4z + 2 = 0 \qquad\qquad \beta : 2x - y + 5z - 2 = 0$$

Let's assume that $P = [p_1, p_2, p_3]$ is some common point of both planes. Since P lies on both planes, its coordinates must satisfy both equations.

$$\begin{aligned} p_1 - p_2 + 4p_3 + 2 &= 0 \\ 2p_1 - p_2 + 5p_3 - 2 &= 0 \end{aligned}$$

We get a system of two equations with three unknowns.

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & -2 \\ 2 & -1 & 5 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 4 & -2 \\ 0 & 1 & -3 & 6 \end{array}\right) \qquad \begin{aligned} p_1 &= 4 - t \\ p_2 &= 6 + 3t \\ p_3 &= t \end{aligned}$$

The system has infinitely many solutions. The two planes have infinitely many common points. They all lie on the line p.

$$p : \begin{aligned} x &= 4 - t \\ y &= 6 + 3t \\ z &= t \end{aligned}$$

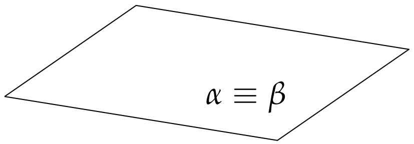
The line p goes through the point $[4,6,0]$ in the direction of the vector $(-1,3,1)$. Verify that both the point and the vector lie in both of the planes.

Note: Taking different steps during elimination might get you a different looking solution, for example

$$p : \begin{aligned} x &= 6 - 2s \\ y &= 6s \\ z &= -2 + 2s \end{aligned}$$

These are also equations of the line p.

93 – Euclidean space $\mathbb{E}_3$, plane – plane intersection



identical planes

Example

$$\alpha:\ 2x+6y+4z+10=0$$
$$\beta:\ 3x+9y+6z+15=0$$

1. method: Compare equations

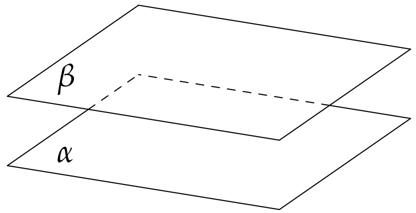
If the equation of the plane α is a multiple of the equation of the plane β, the planes are **identical**.

2. method: Find common points

$$\left(\begin{array}{ccc|c} 2 & 6 & 4 & -10 \\ 3 & 9 & 6 & -15 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\alpha \equiv \beta:\quad \begin{aligned} x &= -5-2t-3s \\ y &= s \\ z &= t \end{aligned}$$

There are many solutions. Since the system has two free variables, the planes are **identical**. The solution corresponds to the parametric equations of the planes.



parallel planes

Example

$$\alpha:\ 2x-6y+4z+10=0$$
$$\beta:\ 3x-9y+6z+10=0$$

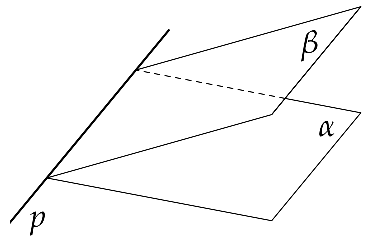
1. method: Compare equations

If the equation of the plane α is a multiple of the equation of the plane β except for the coeficient d, the planes are **parallel**.

2. method: Find common points

$$\left(\begin{array}{ccc|c} 2 & -6 & 4 & -10 \\ 3 & -9 & 6 & -10 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -3 & 2 & -5 \\ 0 & 0 & 0 & -25 \end{array}\right)$$

There is no solution, therefore there are no common points.
Planes are **parallel.**



intersecting planes

Example

$$\alpha:\ 2x+6y+4z-10=0$$
$$\beta:\ 2x+7y+6z-15=0$$

1. method: Compare equations

If the equation of the plane α is not a multiple of the equation of the plane β, the planes are **intersecting**.

2. method: Find common points

$$\left(\begin{array}{ccc|c} 2 & 6 & 4 & 10 \\ 2 & 7 & 6 & 15 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{array}\right)$$

$$p:\quad \begin{aligned} x &= -10+4t \\ y &= 5-2t \\ z &= t \end{aligned}$$

There are many solutions. They all lie on a line, the planes are therefore **intersecting**. The solution corresponds to the parametric equations of this line.

94 – Euclidean space $\mathbb{E}_3$, plane – plane intersection

Exercise

Decide whether the planes α and β are identical, intersecting or parallel (but different).

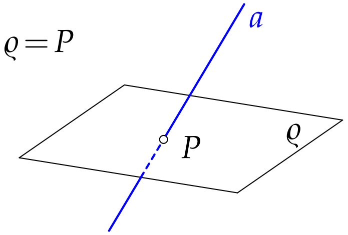
a) $\begin{array}{l} \alpha: \ x-y+2z+2=0 \\ \beta: \ 3x-3y+6z+6=0 \end{array}$

b) $\begin{array}{l} \alpha: \ x-y+2z+2=0 \\ \beta: \ 5x-5y+10z+3=0 \end{array}$

c) $\begin{array}{l} \alpha: \ x-y+2z+2=0 \\ \beta: \ x-3y+4z-4=0 \end{array}$

95 – Euclidean space $\mathbb{E}_3$, plane – plane intersection

$a \cap \varrho = P$



Example

$$x = 2 + 2t$$
$$a : y = 1 - t \qquad \varrho : 3x - 2y - z - 13 = 0$$
$$z = 5t$$

1. method: Compare direction and normal vectors

$(2,-1,5) \cdot (3,-2,-1) \neq 0$ so they are not parallel.

The line cuts through the plane at one single point.

2. method: Find common points

Denote coordinates of their common points $[p_1, p_2, p_3]$. They must satisfy both equations

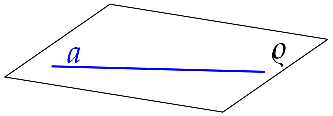
$$p_1 = 2 + 2t$$
$$p_2 = 1 - t \qquad 3p_1 - 2p_2 - p_3 - 13 = 0$$
$$p_3 = 5t$$

Put them together to get a parametr t of the intersection.

$$3(2 + 2t) - 2(1 - t) - 5t - 13 = 0$$
$$t = 3$$

One solution. Therefore **the line cuts through the plane at one single point** $P = [8, -2, 15]$.

$a \in \varrho$



Example

$$x = 4 + t$$
$$a : y = 3t \qquad \varrho : 3x - 2y - z - 13 = 0$$
$$z = -1 - 3t$$

1. method: Compare direction and normal vectors

$(1,3,-3) \cdot (3,-2,-1) = 0$ so they are parallel

Does the point $[4, 0, -1]$ of the line lies in the plane?
$3 \cdot 4 - 2 \cdot 0 + 1 - 13 = 0$ Yes.
The line is embeded in the plane.

2. method: Find common points

Denote coordinates of their common points $[p_1, p_2, p_3]$. They must satisfy both equations

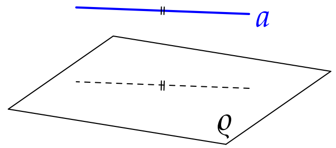
$$p_1 = 4 + t$$
$$p_2 = 3t \qquad 3p_1 - 2p_2 - p_3 - 13 = 0$$
$$p_3 = -1 - 3t$$

Put them together to get a parametr t.

$$3(4 + t) - 2(3t) - (-1 - 3t) - 13 = 0$$
$$0 = 0$$

Many soution. Many common points.
The line is embeded in the plane.

$a \parallel \varrho$



Example

$$x = 2 + 2t$$
$$a : y = 1 + 4t \qquad \varrho : 3x - 2y - z - 13 = 0$$
$$z = 1 - 2t$$

1. method: Compare direction and normal vectors

$(1,\ 3,\ -3) \cdot (3,-2,-1) = 0$ so they are parallel

Does the point $[2, 1, 1]$ of the line lies in the plane?
$3 \cdot 2 - 2 \cdot 2 - 1 - 13 \neq 0$ No.
The line is parallel with the plane but outside it.

2. method: Find common points

Denote coordinates of their common points $[p_1, p_2, p_3]$. They must satisfy both equations

$$p_1 = 2 + 2t$$
$$p_2 = 1 + 4t \qquad 3p_1 - 2p_2 - p_3 - 13 = 0$$
$$p_3 = 1 - 2t$$

Put them together to get a parametr t.

$$3(2 + 2t) - 2(1 + 4t) - (1 - 2t) - 13 = 0$$
$$-10 \neq 0$$

No solution, no common points.
The line is parallel with the plane but outside it.

96 – Euclidean space $\mathbb{E}_3$, plane – line intersection

Exercise

Decide whether the lines a, b, c cut through, are embeded or are parallel with the plane $\varrho : 3x - y + z + 2 = 0$.

$$a : \begin{aligned} x &= 5 - t \\ y &= -1 + 2t \\ z &= -2 + t \end{aligned} \qquad b : \begin{aligned} x &= t \\ y &= 1 + 4t \\ z &= -1 + t \end{aligned} \qquad c : \begin{aligned} x &= 4 + t \\ y &= 2 + t \\ z &= 7 - 2t \end{aligned}$$

97 – Euclidean space $\mathbb{E}_3$, plane – line intersection

Exercise

Decide whether the line a cuts through, is embeded or is parallel with the plane ϱ.

$$a: \begin{aligned} x &= -t - 2 \\ y &= -2t + 4 \\ z &= -2t - 1 \end{aligned} \qquad \varrho: \begin{aligned} x &= -2 + 3s + r \\ y &= 4 + 2s + 2r \\ z &= 1 + 2r \end{aligned}$$

Worksheets for Mathematics I

Introduction to Calculus

Jan Kotůlek

99 – Velocity and distance

We begin with a discussion of the two related problems that motivated the invention of calculus.
Let us look at the relation between the speedometer and the odometer, which is familiar to every driver. The first one measures velocity v and the other one the distance s travelled. Notice the difference in units: s is given in kilometers (or meters) and v in kilometers per hour (or meters per second). Even though we measure both at the time t, a unit of time enters directly only the velocity, not the distance.

The relation between v and s.
Can we find v if we know s? How? And vice versa, if we have the record of the velocity over the time, can we compute the distance traveled? In other words, can we recover missing information of odometer form the complete recors of speedometer?

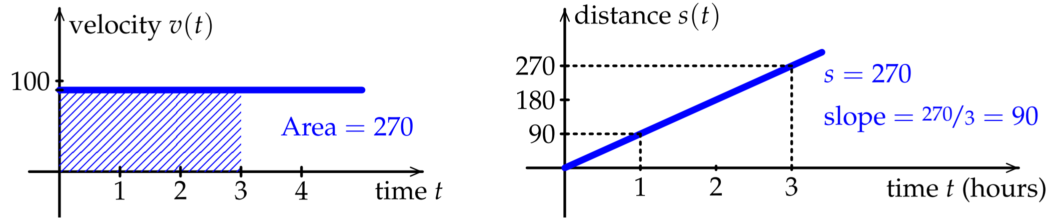
The problem of finding velocity from a record of distance is called **differentiation**, finding distance traveled from the velocity is called **integration**.

Example

Constant velocity
Suppose we travel with fixed velocity $v = 90$ (km per hour). Then s incerases at this constant rate. After an hour the distance is $s = 90$ km, after three hours $s = 270$ and after t hours $s = 90t$. The distance incerases **linearly** with time and its graph is a line with slope 90. The graph of velocity is a horizontal line.
This simple relation of v, s, t needs just algebra:

$$s = v \cdot t$$

Notice that in this example the car starts at full speed and the distance starts at zero, i.e. it is a brand new car.



Constant velocity and linearly increasing distance

Conversely, if s increases linearly, v is constant. The division by time gives the slope. At any point, the ratio s/t is 90. Geometrically, the velocity is **the slope** of the distance graph

$$\text{slope} = \frac{\text{change in distance}}{\text{change in time}} = \frac{v \cdot t}{t} = v$$

Now, how do we compute s from v? Let us start with the graph v and discover the graph of $s = v \cdot t$. The graph of s is given by **the area** under the velocity graph. When v is constant, we got a rectangle with height v and width t, hence its area is v times t. Finding area is called **integration**.

- The slope of distance graph gives the velocity v.
- The area under the velocity graph gives the distance s.

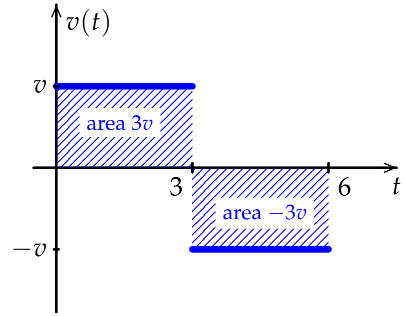
100 – Velocity vs. distance: slope vs. area

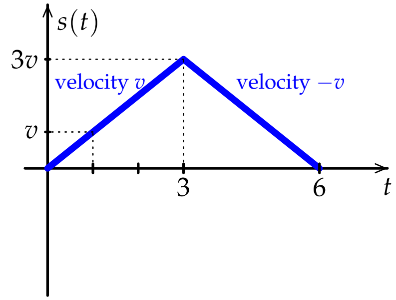
The whole point of calculus is to deal with **non-constant velocities**.

Example

The car goes forward with velocity v for three hours and then returns back where it started with the same speed.

More precisely, the velocity during the return is $-v$, the time needed for return is also $t = 3$, the total distance traveled after the trip will be $s = 0$.





Velocities v and $-v$ give motion there and back again, ending at $s(6) = 0$.

We note that

- the total area under the velocity graph is zero.
- Negative velocity causes the distance graph go downward, the car is moving backward.
- Area below the t axis in the velocity graph is counted as negative.

The number $v(t)$, we say "v of t" is the value of the function v at the time t.
The time t is the **input** to the function, the velocity $v(t)$ at that time is the **output**.

It is easy to write down a formula for our function:

$$v(t) = \begin{cases} +v & \text{if } 0 < t < 3 \\ ? & \text{if } t = 3 \\ -v & \text{if } 3 < t < 6 \end{cases}$$

This function is discontinuous because of the jump speedometer makes at $t = 3$. At that instant of time, velocity is not defined. There is no $v(3)$! You may think that it might be zero, but that leads to troubles. We can't give slope here.
The principle behind the function s is the same: $s(t)$ is the distance at time t and instructions change at $t = 3$:

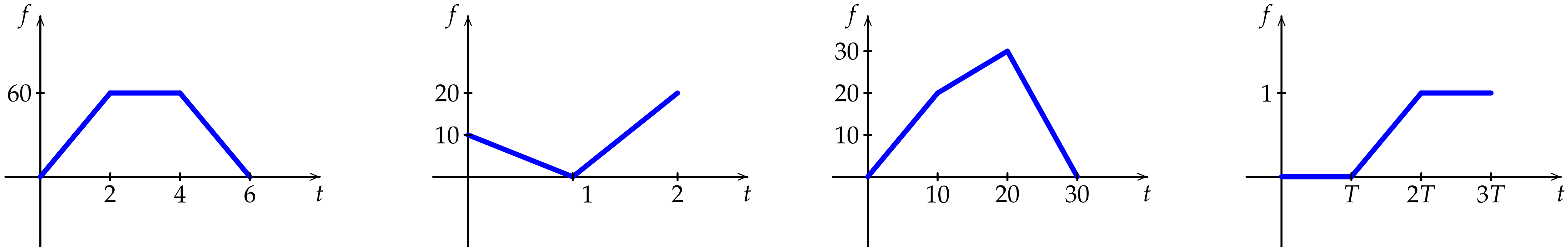
$$s(t) = \begin{cases} v \cdot t & \text{if } 0 \le t \le 3 \\ v(6-t) & \text{if } 3 \le t \le 6 \end{cases}$$

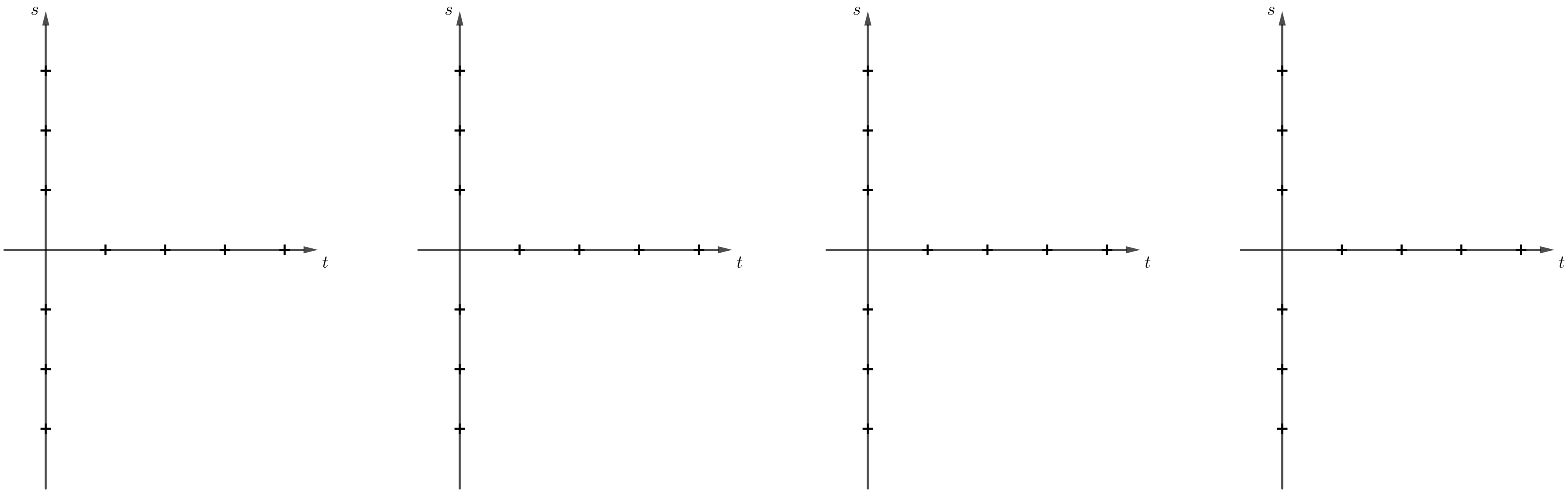
At the switching time, the right-hand-side gives two instructions (one on each line). This would be mistake unless they match: $s(3) = 3v$. Hence, the distance function is continuous.

101 – Velocity vs. distance

Exercise

Draw the distance graphs that correspond to the following velocity graphs.



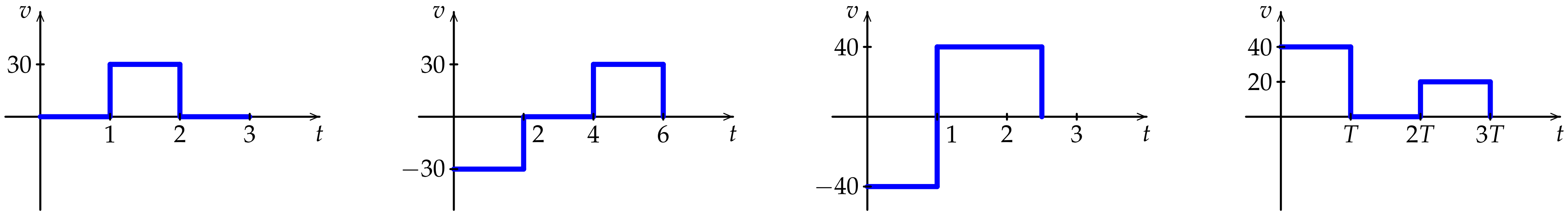


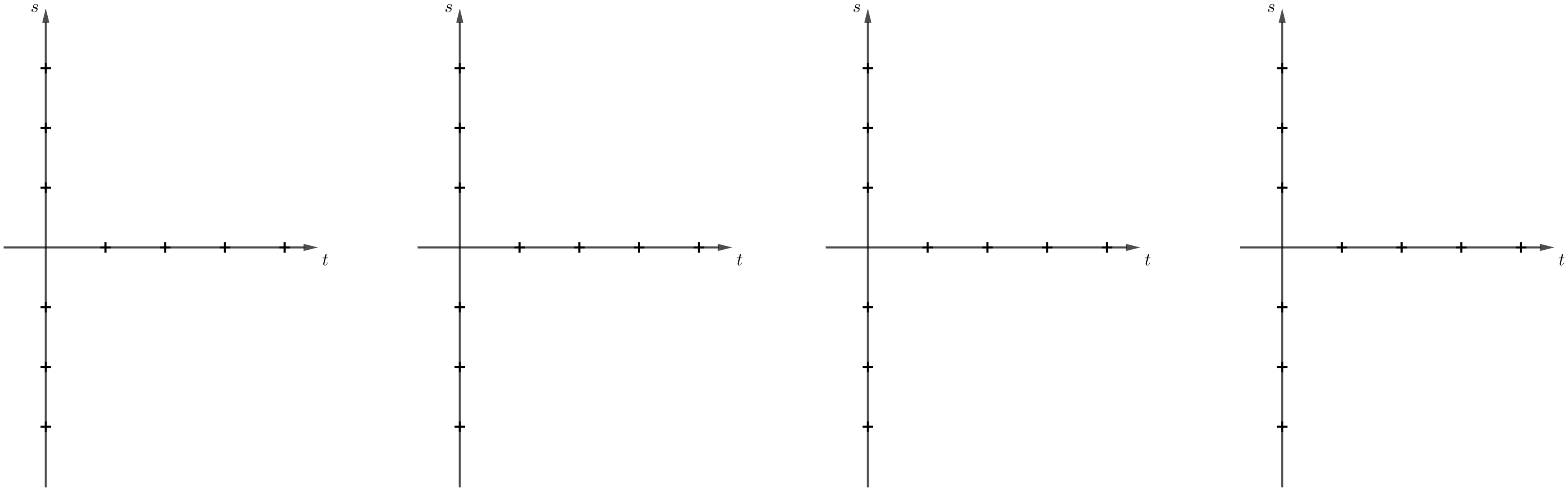
Worksheets for Mathematics I

102 – Distance vs. velocity

Exercise

Draw the distance graphs that correspond to the following velocity graphs. Start from $s = 0$ at $t = 0$ and mark the distance.





103 – Definition of a function

In some way, functions are instructions telling us how to find s at time t. The instructions can be given in the form of

- **explicit formula** $s = f(t)$, e.g. $s = 2t$,
- **implicit equation** $f(x,y) = 0$, e.g. $x + y - 1 = 0$,
- **parametric equations** $x = x(t)$, $y = y(t)$, with $t \in I \subset \mathbb{R}$, describing coordinates of the point in the plane at the time t, e.g.

$$x(t) = 3 + 3t,$$
$$y(t) = 3 - 3t, \quad t \in \langle 0, 1 \rangle.$$

- **table**

t	0	1	2	3	4	5	6
$s(t)$	0	90	180	270	180	90	0

- **graph**, etc.

In practice, the number $f(t)$ is produced from the number t by reading a graph or display of a measuring device, plugging into a formula, solving an equation, or running a computer program.

Definition

The **function** is a rule that assigns one member of the range to each member of the domain. Equivalently, we say that a function is a set of ordered pairs $(t, f(t))$ with no t appearing more than once.
The **domain** of the function f is the set D of inputs, $D \subset \mathbb{R}$.
The **range** of the function f is the set I of outputs, $I \subset \mathbb{R}$. We also say that I is **image** of D, $I = f(D)$.

The input t is mapped to the output $s(t)$, which changes as t changes. All calculus is about the **rate of change**. This rate was our function $v(t)$.

Functions are used to build **deterministic models**, i.e. models which always produce the same output from a given starting condition or initial state. Such models are widely used in mathematics, e.g. systems in chaos theory are deteministic, nevertheless strongly dependent on the initial conditions, in physics, where the laws described by differential equation (Newton law, Schrödinger equation, etc.), even though the state of the system at a given time is often difficult to describe explicitly, or in computer science, e.g. Turing machine is deterministic.

104 – The velocity changing at an instant

There are two central questions leading in opposite directions that calculus was invented to solve:
1) If the velocity is changing, *how can we compute the distance travelled*?
2) If the graph $s(t)$ of the distance is not a straight line, *what is its slope*?

The first step is to let the velocity change in the steadiest possible way:

Example

Suppose that $v(t) = 2t$ is the velocity at each time t. Find the distance $s(t)$.

To describe the situation with our driver example, a physicist would say that the driver steps on the gas, the spedometer goes steadily up and the acceleration is constant (it equals 2).
We measure t in seconds and v in meters per second. After 5 seconds the speed is 10 m/s after 12, 5 seconds the speed is 25 m/s, which is 90 km/h. The acceleration is clear. Actually, it is the slope of the velocity graph. *But how far has the car gone*?

Example

Suppose that $s(t) = t^2$ is the distance traveled by time t. Find the velocity $v(t)$.

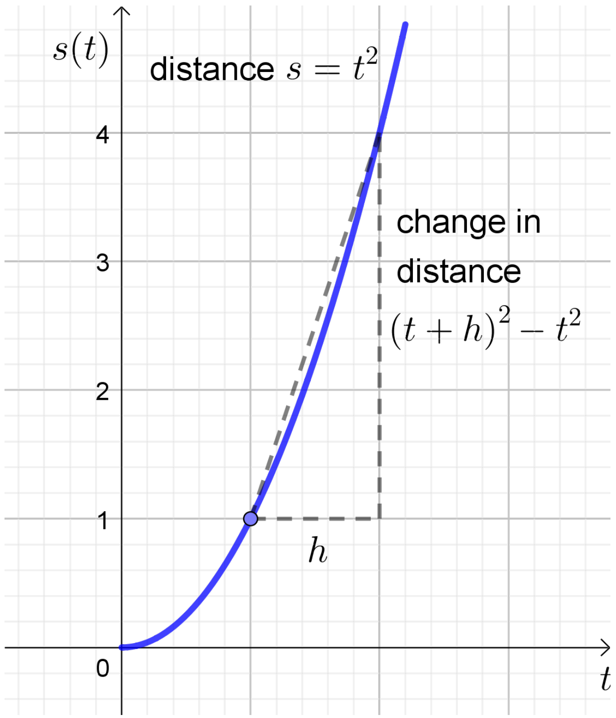
The distance graph of $s(t)$ is a **parabola**. It starts at zero, the car is new. At $t = 3$ the distance is $s = 9$, at $t = 5$, the distance is $s = 25$ and by $t = 10$, s reaches 100.
Velocity is distance divided by time, but what happens when the speed is changing? Dividing $s = 100$ by $t = 10$ gives $v = 10$, this is the **average velocity** over the first ten seconds. But how do we find the **instantaneous velocity** without looking at the speedometer at the exact instant when $t = 10$?
The problem is that the distance is not distributed evenly. As the car goes faster, the graph of t^2 gets steeper and more distance is covered in every following second. We can try an approximation. The average velocity between $t = 10$ and $t = 11$ may be a good approximation to the speed at the moment $t = 10$ and averages are easy to find:

$$\text{average } v = \frac{\text{change in } s}{\text{change in } t} = \frac{s(11) - s(10))}{11 - 10} = \frac{121 - 100}{1} = 21.$$

The car covered 21 meters in that second and its average speed was 21 m/s. Since it was still gaining speed, the velocity at the beginning $t = 10$ was below 21.

105 – The velocity changing at an instant

What is the average geometrically? It is a slope. But not the slope of the curve. *The average velocity is the slope of a straight line* joining two point on the curve. Thus, we pretend the velocity is constant and we are back in the previous easy case.



The graph of quadratic distance function $s(t) = t^2$ and its velocity.

We can also find the average over a smaller time interval, for example half-second between $t = 10.0$ and $t = 10.5$. We again divide the change in distance by the change in time:

$$\frac{s(10.5) - s(10.0))}{10.5 - 10} = \frac{110.25 - 100}{0.5} = 20.5.$$

This is closer to the speed at $t = 10$, but still not exact. The way to find $v(10)$ is to proceed with *reducing the time interval*.

Finding the slope between points that are closer and closer on the curve is the key to the differentail calculus. The "limit" is the slope at a single point.

We can compute the average velocity between $t = 10$ and any later time $t = 10 + h$ by the same algebra:

$$v_{\text{av}} = \frac{(10+h)^2 - 10^2}{h} = \frac{100 + 20h + h^2 - 100}{h} = 20 + h.$$

This agrees with our previous calculations: for the time interval from $t = 10$ to $t = 11$ we had $h = 1$ and the average was $20 + h = 21$, for the half-second we had $h = \frac{1}{2}$ and the average was $20 + \frac{1}{2} = 20.5$. Over a milionth of a second the average would be 20.000 001, which is very close to 20.

We are ready to show that for the distance function $s(t) = t^2$, the velocity function $v(t) = 2t$. This is the key computation of calculus: we compute the distance at $t + h$, subtract the distance at t and divide by h:

$$v_{\text{av}} = \frac{s(t+h) - s(t)}{h} = \frac{(t+h)^2 - t^2}{h} = \frac{t^2 + 2th + h^2 - t^2}{h} = 2t + h.$$

As h approaches zero, the average velocity for the distance function $s(t) = t^2$ approaches $v = 2t$.

Worksheets for Mathematics I

Computation of the derivative

Jan Kotůlek

107 – The derivative of a function

Now we can give the formal definition of the derivative. It generalizes the examples form the previous chapter. We introduce the derivative of the function $f(x)$ and use the new symbols f' and df/dx for it.

Definition

The **derivative** $f'(x)$ or df/dx is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The ratio on the right-hand side resembles the average velocity over a short time Δt. The derivative is its limit as the time interval Δt approaches zero.
Look carefully at each part: The $f(x+\Delta x)$ is the value of f at $x+\Delta x$, the $f(x)$ is the value at x and the subtraction gives **change in value**, often denoted by Δf. The derivative is the ratio $\Delta f/\Delta x$, change in value divided by change in argument, in agreement with our previous examples.
The limit of the average velocity is the derivative,

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t},$$

if this limit exists. Behind the innocent word "limit", there is a whole process of approaching. Its understanding requires working with the concept of infinity and this is why the general theory of limits is not particularly simple. Here it is sufficient to understand it within the scope presented in the previous chapter. As it is not that hard either, we work on it further it in the next chapter, where we also formulate an efficient method for computation of limits, l'Hôpital rule.
The left-hand sides $f'(t)$ and df/dt are **instantaneous speed**. They give the slope at the instant t.

The notation hides two things worth mentioning.

First, the time step can be *negative*. We compute average $\Delta f/\Delta t$ over a time interval *before the time t*, instead of after. This ratio also approaches df/dt.
Second, *the derivative might not exist*. The averages $\Delta f/\Delta t$ might not approach a limit (the same one for the time running forwards and backwards). In that case $f'(t)$ is not defined.

108 – The derivative of a function

Example

Review the calculation of the instant velocity for the distance function $f(t) = t^2$:

$$\begin{aligned}\frac{\Delta f}{\Delta t} &= \frac{f(t+\Delta t) - f(t)}{\Delta t} \\ &= \frac{t^2 + 2t\Delta t + (\Delta t)^2 - t^2}{\Delta t} \\ &= \frac{2t\Delta t + (\Delta t)^2}{\Delta t} \\ &= 2t + \Delta t\end{aligned}$$

Note that we take these steps before Δt goes to zero. If we set $\Delta t \to 0$ too early, we learn nothing, as the ratio becomes $0/0$, an expression which does not have meaning so far. The theory of limits will later allow us to understand it.
The numbers Δf and Δt must approach zero together, not separately. Then, their ratio $2t + \Delta t$ gives the correct average speed.

Example

Constant velocity $v = 3$ up to time $t = 5$, then stop.

For $t \in (0, 5)$ we have $f'(t) = 3$ and thus $f(t) = 3t$. After stopping time the distance remains fixed at $f(t) = 15$ and its graph is flat beyond the time $t = 5$. Hence, $f(t + \Delta t) = f(t)$ and $\Delta f = 0$. This means that

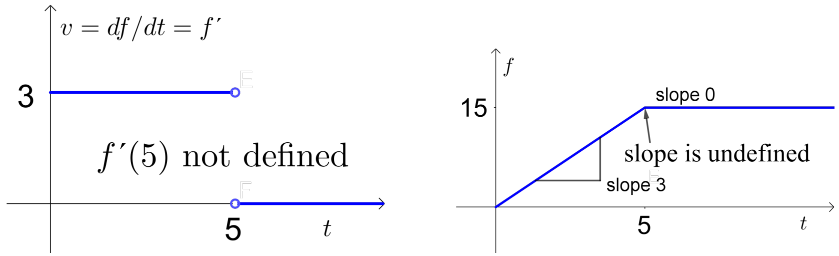
Theorem

The derivative of a constant function is zero.

Indeed, it holds for $t > 5$:

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{0}{\Delta t} = 0.$$

The derivative is not defined for $t = 5$, as the velocity falls suddenly from 3 to zero. The ratio $\Delta f / \Delta t$ depends at this moment on whether Δt is positive or negative.



The graphs for $f'(t)$ and $f(t)$ for the instant stop at $t = 5$.

109 – The derivative of power functions

So far, our variable was the time t. In many textbooks, the general mathematical unknown x is used. Nevertheless, the name of the variable does not play any role. In the next example, we use the price p as variable.
Similarly, f is not the only possibility. It can't be used for every function. The letter f is useful, as it stands for the word function, but we are free to choose $y(x)$ or, in our case, $d(p)$.

Example

Consider the **demand function** $d(t) = 1/p$. The meaning is clear, increasing price p reduces the demand $d(p)$.

How quickly does $1/p$ change when p changes? It will be clear when we find the derivative of $1/p$ for all p.
First, take Δd and simplify to

$$\Delta d = \frac{1}{p+\Delta p} - \frac{1}{p} = \frac{p-(p+\Delta p)}{p(p+\Delta p)} = \frac{-\Delta p}{p(p+\Delta p)}.$$

Dividing by Δp and letting $\Delta p \to 0$ gives

$$\begin{aligned} \frac{\Delta d}{\Delta p} &= \frac{\frac{-\Delta p}{p(p+\Delta p)}}{\Delta p} = \frac{-1}{p(p+\Delta p)} \\ d'(p) &= \lim_{\Delta p \to 0} \frac{-1}{p(p+\Delta p)} = \frac{-1}{p^2 + p\Delta p} = \frac{-1}{p^2} \end{aligned}$$

Check the algebra for $p = 2$ and $\Delta p = 1$. The demand $1/p$ drops from $\frac{1}{2}$ to $\frac{1}{3}$ at $p + \Delta p$. The difference of demand is $\Delta d = -1/6$, which agrees with $-1/(2 \cdot 3)$ in the first line. As the steps Δp and Δd get smaller, their ratio approaches $-1/(2^2) = -1/4$.

The slope is also a function. The whole calculus is about two functions, $y = f(x)$ and $y' = df/dx$.

The derivative of the function $y = 1/x$ is

$$\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$$

110 – The derivative of powers and polynomials

We have already computed the derivatives of $y = x^2$ and $y = 1/x = x^{-1}$. Now, we show that the derivatives of all functions of the form $y = x^n$ follow the same pattern.

Theorem

The derivative of the nth power is given by

$$(x^n)' = n \cdot x^{n-1}, \qquad \text{for all } n \in \mathbb{R} \setminus \{0\}$$

The exception $n = 0$ is the constant function $y = x^0 = 1$. Its derivative, as we've already discovered, is zero.

Example

Using the previous theorem, compute the derivatives of the quadratic function x^2, the linear rational function $1/x$ and the square root $\sqrt{x}$.

$$\begin{aligned}
(x^2)' &= 2x^1 = 2x, \\
\left(\frac{1}{x}\right)' &= (-1) \cdot x^{-2} = -\frac{1}{x^2} \\
(\sqrt{x})' &= \left(x^{\frac{1}{2}}\right)' = \frac{1}{2} \cdot x^{-1/2} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Exercise

Compute the derivatives: $\left(x^5\right)'$, $\left(\frac{1}{p^3}\right)'$, $\left(\sqrt[3]{t^2}\right)'$, $\left(\frac{1}{\sqrt[5]{r^9}}\right)'$.

Remark

The pattern follows from the definition of the derivative and **binomial expansion** of $(x+h)^n = x^n + nx^{n-1}h + \cdots + nxh^{n-1} + h^n$. Every element is of the form $\binom{n}{i}x^{n-i}h^i$ for $i \in \{0, \ldots, n\}$ and the coefficients follow from the Pascal triangle.
Due to the standard procedure from the definition, the only elements that matter are the ones with exactly one h.

Exercise

Work out the cases $y = x^3$ and $y = x^4$, i.e. $n = 3, 4$, in detail (from the definition).

111 – Linearity of the derivative

A huge number of functions are **linear combinations** like $f(x) = x^2 + x$ or $f(x) = x^2 - x$, or $f(x) = 5x^2$ or $f(x) = x/2$. In general also all of it at once: $f(x) = 5x^2 - \frac{1}{2}x + \sqrt{3}$. You've met such linear combinations in detail in the chapter on linear algebra.
If we need to add or subtract or multiply by 5 or divide by 2, we can *do the same with the derivatives*.

Theorem

The derivative is linear, i.e., the following holds:

1. $(c \cdot f(x))' = c \cdot f'(x)$ for any constant $c \in \mathbb{R}$.
2. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.

Example

We show the rule for a polynomial, in our case the quadratic function $y = 3x^2 - 4x + 5$, but the rules allow any combination of f and g.

$$(3x^2 - 4x + 5)' = 3(x^2)' - 4(x)' + (5)' = 3 \cdot (2x) - 4 \cdot 1 + 0 = 6x - 4.$$

Exercise

Compute the derivatives:

$$\left(4x^5 - \frac{3}{x^6}\right)', \qquad \left(\sqrt[3]{8 \cdot x^2} + \frac{10}{\sqrt[5]{x^9}} - 3\sqrt{4}\right)'.$$

112 – The derivative of product and quotient

We want to compute the derivative of multiplication of two functions. It is different from multiplication of function by a constant, as we can see from the following baby example:

$$\begin{aligned}(x^2\cdot x)' &\overset{?}{=} (x^2)'\cdot x' = 2x\cdot 1 = 2x,\\ (x^2\cdot x)' &= (x^{2+1})' = (x^3)' = 3x^2.\end{aligned}$$

It follows that $(f(x)\cdot g(x))' \neq f'(x)\cdot g'(x)$.
The correct formula is named after one of the inventors of the calculus Gottfried Wilhelm Leibniz (1646–1716).

Theorem (Leibniz product rule)

For any differetiable functions f and g the following holds:

$$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x).$$

Example

Compute the derivative of the function $y = (x^2+4x-6)\sqrt{x}$

$$\begin{aligned} y' &= (x^2+4x-6)'\sqrt{x} + (x^2+4x-6)(\sqrt{x})'\\ &= (2x+4)\sqrt{x} + (x^2+4x-6)\cdot\frac{1}{2\sqrt{x}}.\end{aligned}$$

The formula for the quotient of two functions contains an expression similar to Leibniz rule in the numerator:

Theorem (Quotient rule)

For any differetiable functions f and g the following holds:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}.$$

Exercise

Compute the derivative of

$$y = \frac{(\sqrt{x}-3)^2}{x}$$

113 – The derivative of composed function

Example

If the function $u(x)$ has slope du/dx, determine the slope of the composed function $f(x) = (u(x))^2$.

The first observation, e.g. with $u(x) = x^2$, gives the function $f(x) = x^4$, for which $(du/dx)^2 = (2x)^2$. On the other hand $f' = (x^4)' = 4x^3$. Hence, *the derivative of u^2 is not $(du/dx)^2$.*
To get the correct answer, we have to start with $\Delta f = f(x + \Delta x) - f(x)$:

$$\Delta f = (u(x + \Delta x))^2 - (u(x))^2 = [u(x + \Delta x) + u(x)] \cdot [u(x + \Delta x) - u(x)]$$

due to factorization $a^2 - b^2 = (a + b)(a - b)$. Notice we don't have $(\Delta u)^2$.
Now we divide the Δf, the change in u^2, by Δx

$$\frac{\Delta f}{\Delta x} = [u(x + \Delta x) + u(x)] \cdot \frac{[u(x + \Delta x) - u(x)]}{\Delta x},$$

where the second term is just du/dx. Taking the limit we get

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = 2u(x) \cdot \frac{du}{dx}.$$

Example

Compute the derivative of the function $f(x) = (\sqrt{x} - 1)^2$.

In agreement with the previous calculation we get:

$$f'(x) = 2 \cdot (\sqrt{x} - 1) \cdot \frac{1}{2\sqrt{x}} = 1 - \frac{1}{\sqrt{x}}.$$

Let us check the answer by computing the derivative without the rule. Factorig the square we get $(\sqrt{x} - 1)^2 = x - 2\sqrt{x} + 1$. In this form we can compute using the rule for nth power and confirm the result.

Notice the the result is the product of the derivative of "outer" function, the second power, and the inner function u. Let us state the rule in the general form:

Theorem (Chain rule)

For any differetiable functions f and g it holds:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Exercise

Compute the derivative of the composed functions $y = (x^3 - 1)^4$ and $y = \sqrt{8 - 4x - 2x^2}$.

114 – The derivatives of exponential and logarithmic functions

We state now the rules for the derivatives of exponential functions $y = a^x$ and logarithmic functions $y = \log_b(x)$, where a and b are called **base** and both $a, b \in (0,1) \cup (1,\infty)$. For the same base, the functions $y = a^x$ and $y = \log_a(x)$ are mutually inverse, so it holds

$$a^{\log_a x} = x \qquad \text{and} \qquad \log_a(a^x) = x.$$

The rules for the derivative of the functions $y = \mathrm{e}^x$ and $y = \ln(x)$, with the Euler number $\mathrm{e} = 2.71\ldots$ as a base, are particularly simple.

Theorem

For the exponential function $y = \mathrm{e}^x$ it holds

$$(\mathrm{e}^x)' = (\mathrm{e}^x)\,.$$

For a general exponential function $y = a^x$ a factor should be added. Let us show its value using the formula of the inverse:

$$(a^x)' = \left(\mathrm{e}^{\ln(a^x)}\right)' = \left(\mathrm{e}^{x\cdot(\ln a)}\right)' = \mathrm{e}^{(\ln a)\cdot x} \cdot ((\ln a)\cdot x)' = a^x \cdot (\ln a)$$

For the composed function $y = \mathrm{e}^{u(x)}$, with an inner function $u(x)$, the chain rule gives:

$$\left(\mathrm{e}^{u(x)}\right)' = \left(\mathrm{e}^{u(x)}\right) \cdot u'(x).$$

Example

Compute the derivative of the functions $y = \mathrm{e}^{\sqrt{x}}$ and $y = 2^{3t+1}$.

$$\begin{aligned} \left(\mathrm{e}^{\sqrt{x}}\right)' &= \mathrm{e}^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \\ \left(2^{3t+1}\right)' &= 2^{3t+1} \cdot (\ln 2) \cdot 3 \end{aligned}$$

Theorem

For the logarithmic function $y = \ln(x)$ it holds

$$(\ln(x))' = \frac{1}{x}$$

Similarly as above, for a general logarithmic function $y = \log_a(x)$ a factor should be added:

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a} \cdot (\ln x)' = \frac{1}{\ln a} \cdot \frac{1}{x}$$

For the composed function $y = \ln(u(x))$, with an inner function $u(x)$, the chain rule gives:

$$(\ln(u(x)))' = \frac{1}{u(x)} \cdot u'(x).$$

Example

Compute the derivative of the functions $y = \ln(x^2 + 4x + 5)$ and $y = \log_2(3x)$.

$$\begin{aligned} \left(\ln(x^2+4x+5)\right)' &= \frac{1}{x^2+4x+5} \cdot (2x+4) \\ (\log_2(3x))' &= \frac{1}{\ln 2} \cdot \frac{1}{3x} \cdot 3 = \frac{1}{x \cdot \ln 2} \end{aligned}$$

115 – The derivatives of the trigonometric functions

The sine and cosine functions are important for description of *oscillations*. It is a beautiful fact that for $y = \sin(x)$ is $y' = \cos(x)$. We derive the derivative by the standard limit technique:

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\Delta y}{\Delta x} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

This looks harder than x^n, as we need so called **addition formula**

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

Since we are going to look on what happens for $h \to 0$, we factor out the $\sin(x)$ and $\cos(x)$ and get

$$\lim_{h\to 0} \frac{\Delta y}{\Delta x} = \sin(x) \left(\lim_{h\to 0} \frac{\cos h - 1}{h} \right) + \cos(x) \left(\lim_{h\to 0} \frac{\sin h}{h} \right).$$

It is no longer easy to divide by h. We proceed with showing the value of the two limits without proof, which we provide in the next chapter

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \qquad \text{and} \qquad \lim_{h\to 0} \frac{\sin h}{h} = 1.$$

Therefore, we arrive at

$$\frac{dy}{dx} = \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos x.$$

Theorem

For the derivatives of sine and cosine functions it holds:

$$(\sin(x))' = \cos(x) \qquad (\cos(x))' = -\sin(x)$$

Note the sign in the second formula, which stems from the addition formula for the cosine, $\cos(x+h) = \cos x \cos h - \sin x \sin h$.

Example

Compute the derivatives of

a) $y = 4 + \sin(2t+1)$

b) $y = 3\cos\left(\frac{t}{2} + \frac{\pi}{6}\right)$

c) $y = \tan^2(5\omega)$

We first note that

$$(\sin(u(x)))' = \cos(u(x)) \cdot u'(x).$$

Therefore, it holds

$$\begin{aligned} &\text{a)} \quad (4 + \sin(2t+1))' = \cos(2t+1) \cdot 2, \\ &\text{b)} \quad \left(3\cos\left(\frac{t}{2} + \frac{\pi}{6}\right)\right)' = 3 \cdot \left(-\sin\left(\frac{t}{2} + \frac{\pi}{6}\right)\right) \cdot \frac{1}{2}. \end{aligned}$$

To solve c) we first deduce the formula for the derivative of the tangent function from the quotient rule:

$$(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$

Therefore,

$$\left(\tan^2(5\omega)\right)' = 2 \cdot \tan(5\omega) \cdot \frac{1}{\cos^2(5\omega)} \cdot 5.$$

Exercise

Compute the derivative of $y = \cot(2x)$.

116 – The derivatives of the inverse trigonometric functions

For the sake of completeness we state the formulas for the derivatives of the inverse trigonometric functions (occasionally also called arcus functions or cyclometric functions.

Theorem

The derivatives of the inverse trigonometric functions it holds:

$$\begin{aligned}(\arcsin(x))' &= \frac{1}{\sqrt{1-x^2}}\\ (\arccos(x))' &= -\frac{1}{\sqrt{1-x^2}}\\ (\arctan(x))' &= \frac{1}{1+x^2}\end{aligned}$$

Example

Compute the derivatives of

a) $y = \arcsin\sqrt{x}$

b) $y = \arctan\frac{1}{x}$

As it holds

$$\begin{aligned}(\arcsin(f(x)))' &= \frac{1}{\sqrt{1-(f(x))^2}}\cdot f'(x),\\ (\arctan(f(x)))' &= \frac{1}{1+(f(x))^2}\cdot f'(x),\end{aligned}$$

we get:

a) $(\arcsin(\sqrt{x}))' = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}},$

a) $\left(\arctan\left(\frac{1}{x}\right)\right)' = \frac{1}{1+(\frac{1}{x})^2}\cdot\left(\frac{-1}{x^2}\right) = \frac{-1}{x^2+1}.$

117 – Overview of the necessary formulas

$$(c \cdot f(x))' = c \cdot f'(x) \text{ for any constant } c \in \mathbb{R},$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x),$$

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x),$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2},$$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

$$(x^n)' = n \cdot x^{n-1}, \qquad \text{for all } n \in \mathbb{R} \setminus \{0\},$$

$$(\mathrm{e}^x)' = (\mathrm{e}^x),$$

$$(\ln(x))' = \frac{1}{x},$$

$$(\sin(x))' = \cos(x),$$

$$(\cos(x))' = -\sin(x),$$

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan(x))' = \frac{1}{1+x^2}$$

118 – The derivative of a function

Exercise

Compute the derivatives of the following functions:

a) $y = 2 \cdot x^2 \cdot \sqrt{x}^3 + \sqrt[3]{8x^4} - \dfrac{1}{2x^3} + \dfrac{\sqrt{2}}{2}$, for $x_0 = 1$

b) $M_{II}(x) = F \cdot (a + x) - q_2 \cdot \dfrac{a}{2} \cdot \left(\dfrac{3}{4}a + x\right) - q_2 \cdot x \cdot \dfrac{x}{2}$, for $x_0 = 2$

c) $y = \dfrac{\tan x}{\sin(2x)}$, for $x_0 = \frac{\pi}{3}$

d) $y = \cos(1) - 3 \cdot \cos^2\left(\dfrac{x}{3} - \dfrac{\pi}{6}\right)$, for $x_0 = 0$

119 – The derivative of a function

Exercise

Compute the derivatives of the following functions with respect to their independent variables:

a) $N(y) = \sqrt{\dfrac{\beta + x}{1 - y}}$, for $y_0 = 0$

b) $F(u) = \dfrac{\arcsin(1 - 4u)}{2}$, for $u_0 = \frac{1}{4}$

c) $G(z) = \ln \dfrac{4}{2z - 4} - \ln 8$, for $z_0 = 3$

d) $A(t) = A_0 + 3\mathrm{e}^{-2\alpha t + t_0}$, for $t_0 = 0$

e) $V(r) = \sqrt{\dfrac{\pi p r^4}{8\eta\ell}}$, for $r_0 = \frac{1}{2}$

Worksheets for Mathematics I

Applications of the derivative

Jan Kotůlek

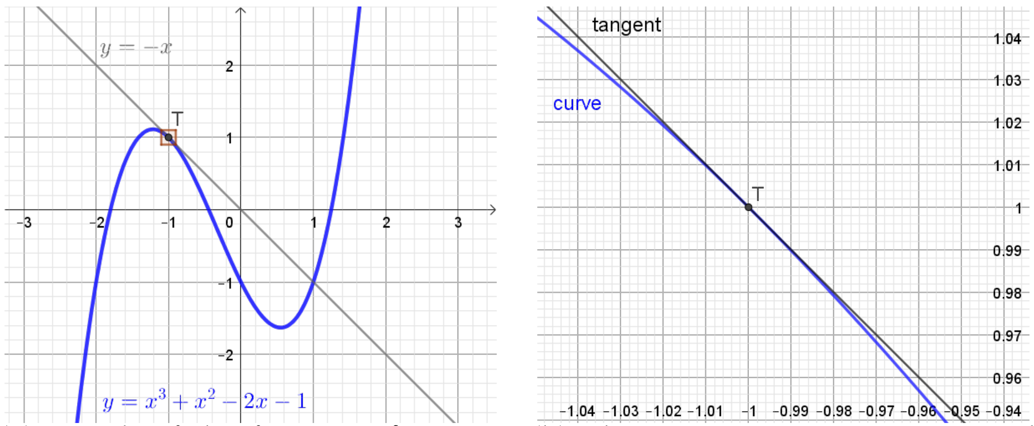
121 – Tangent line

If we focus our attention near a single point, on a very short range, a curve looks straight. Looking through microscope, or zooming in a computer program, the graph becomes nearly linear.

Example

Consider the function $f : y = x^3 + x^2 - 2x - 1$. At the point $x_0 = -1$, the value of f is $y_0 = f(-1) = 1$, which gives the point of tangency $T = [-1, 1]$. The slope of f is given by $dy/dx = 3x^2 + 2x - 2$. At $x = -1$ the slope is $f'(-1) = 3 \cdot (-1)^2 + 2 \cdot (-1) - 2 = -1$.
The equation of the tangent line is $y - 1 = (-1)(x - (-1))$, that is

$$y = -x.$$



(a) Graph of the function $f : y = x^3 + x^2 - 2x - 1$ and its tangent at $x = -1$.

(b) The previous situation zoomed to the interval $I = \langle -1,04; -0,94 \rangle$.

A straight line is determined by two of its points or by its point and slope. That is the situation with the tangent line:

1. The equation of a line has the form $y = kx + q$.
2. The number k is the slope of the line, as $dy/dx = k$.
3. The number q adjusts the line to go through the required point of tangency.

The curve and its tangent line have the same slope at the point of tangency.

Definition

Equation of the **tangent line** t to f at the point T:

$$t : y - f(x_0) = k_t \cdot (x - x_0)$$

for $T = [x_0, f(x_0)]$ and $k_t = f'(x_0)$.

122 – Tangent line

Exercise

Write down the equation for the tangent line t to the graph of the following function:

a) $y = \dfrac{1}{(2-\mathrm{e}^x)^2}$

b) $y = 2x \cdot \cos\left(\dfrac{x}{2}\right) + 1$

c) $y = \mathrm{e}^{\sin\left(\frac{x}{2}\right)}$

d) $y = \sqrt{5-\mathrm{e}^{-4x}}$

e) $y = \cos\left(\dfrac{3x}{1-2x}\right)$

f) $y = \sqrt{1+2\ln\left(x^2+x+1\right)}$

at the point P_y, intersection point with the coordinate axis y, i.e. the line $x=0$.
Find the slope of t.

Hints

Equation of the tangent line t to f at the point T:
$t: y - f(x_0) = k_t \cdot (x - x_0)$
for $T = [x_0, f(x_0)]$ and $k_t = f'(x_0)$

123 – Tangent line

Exercise

Write down the equations for the tangent lines t_i to the graph of the following function:

a) $y = \dfrac{x+1}{x^2+1}$

b) $y = \sqrt{x^3+1}$

c) $y = \dfrac{\ln\left(x^2-3\right)}{x}$

d) $y = \ln\left(x^2 - x + 1\right)$

e) $y = 1 - \mathrm{e}^{x^2+2x-8}$

f) $y = x \cdot \operatorname{arctan}(x-2)$

at intersection points P_{x_i} with the coordinate axis x, i.e. the line $y = 0$.
Find the slopes of t_i.

Hints

Equation of the tangent line t to f at the point T:
$t : y - f(x_0) = k_t \cdot (x - x_0)$
for $T = [x_0, f(x_0)]$ and $k_t = f'(x_0)$

124 – Normal line

There is another important line, closely connected to the tangent line. It is **perpendicular** to the tangent and to the curve and it passes through the same point of tangency. It is called **normal line**, usually denoted by n.
Let us discuss its slope.
According to the rule that slopes of perpendicular lines multiply to give -1,the following holds:

Theorem

If the tangent has slope m, the normal line has slope $-1/m$.

Light rays follow the direction of the normal line.
Wood fires move perpendicular to the fire line.

Example

Determine the tangent and normal line to the curve $y = x^3 - 2$ at the point of tangency $[2;6]$.

The slope of the tangent line is

$$k_t = y'(2) = \left(3x^2\right)\Big|_{x=2} = 12.$$

Hence, the point-slope equation of the tangent line is

$$t: \quad y - 6 = 12(x-2).$$

As $k_n = -1/k_t$ the point-slope equation of the normal line is

$$n: \quad y - 6 = \frac{-1}{12}(x-2).$$

125 – Taylor polynomial

In the previous section we defined a linear approximation to estimate values of a function f at a neighborhood of the point a with known value. It can be used efficiently if

- we know the value $f(a)$
- we can easily compute the value of the first derivative of f at the point a.

However, this is not always the case. For example, a linear approximation of the Euler number $\mathrm{e} \doteq 2.71\ldots$, i.e. the value of the function $y = \mathrm{e}^x$ at the point $a = 1$, is not sufficient for precise calculations.
We can't use the tangent line at the point $a = 1$, as the value of the function and the derivative is just e.
Hence, we are forced to use the tangent approximation at $a = 0$, which gives us

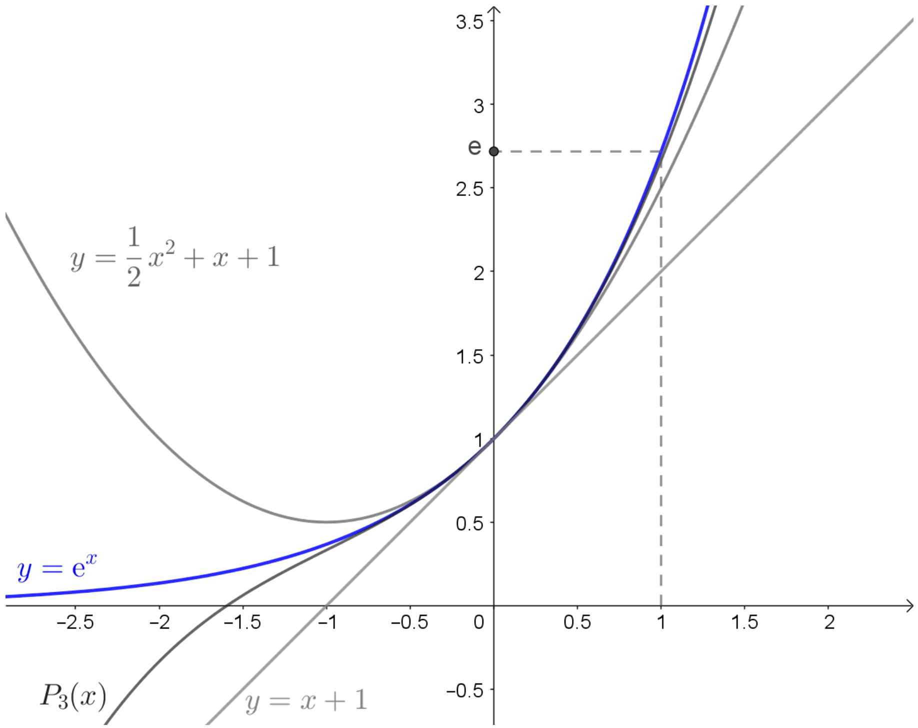
$$\mathrm{e}^1 \approx \mathrm{e}^0 + \mathrm{e}^0(x-0) = 1 + x = 2,$$

which is far from satisfactory.
It is a straightforward idea to approximate the value with a quadratic function, also with the help of the second derivative, which gives us:

$$\mathrm{e} \approx \mathrm{e}^0 + \mathrm{e}^0(x-0) + \frac{\mathrm{e}^0}{2!}(x-0)^2 = \left(1 + x + \frac{x^2}{2}\right)_{x=1} = 2.5.$$

The higher degree polynomials give us required accuracy:

$$\begin{aligned}
\mathrm{e} &\approx \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!}\right)_{x=1} &&= 2 + \tfrac{2}{3} = 2.\bar{6},\\
&\approx \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)_{x=1} &&= 2 + \tfrac{17}{24} = 2.708\bar{3},\\
&\approx \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right)_{x=1} &&= 2 + \tfrac{43}{60} = 2.71\bar{6},
\end{aligned}$$



The polynomial obtained in this way is called **Taylor polynomial**.

126 – Taylor polynomial

Definition

The polynomial of the degree $n \in \mathbb{N}$ of the form

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the **Taylor polynomial** of the order n for the function f centred at the point a.

We can use the Taylor polynomial to approximate the values of f at a neighborhood of a if

- we know the value $f(x_0)$,
- we can easily compute the first n derivatives of the function f at the point a.

Remark

For the centre at the origin, $a = 0$, the polynomial is also called **Maclaurin polynomial** and denoted $M_n(x)$.

127 – Taylor polynomial

Example

For the function $f : y = \cos(2x)$ write down the formula for the Maclaurin polynomial of the order 4.

The formula for the Maclaurin polynomial is easily obtained from $T_n(x)$ by substituting $a = 0$:

$$M_4(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

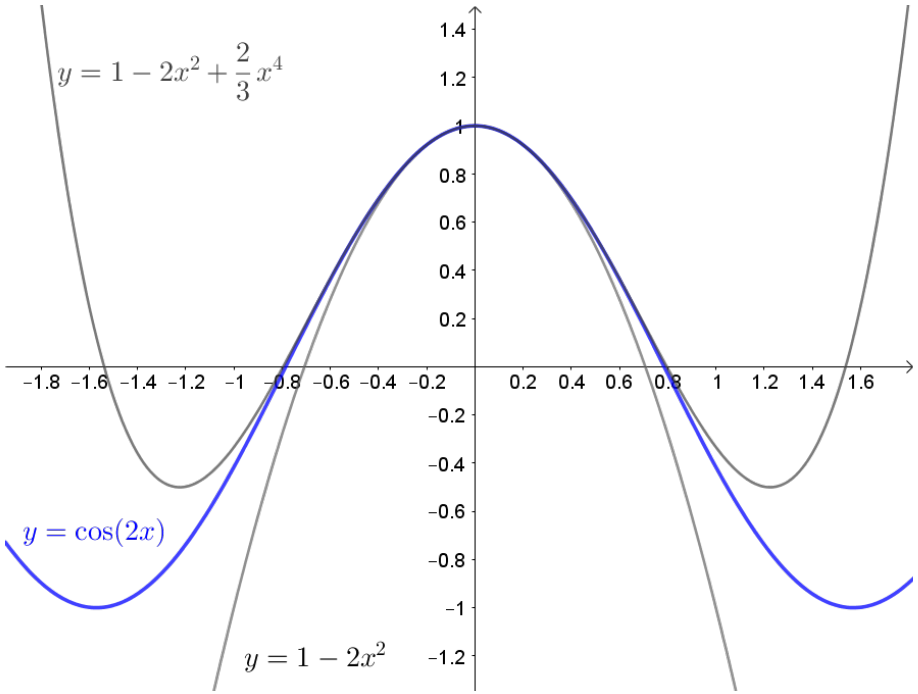
We start with computing the first four derivatives of $f(x) = \cos(2x)$ and moreover we immediately compute their values at $a = 0$. We get:

$$\begin{aligned}
f(x) &= \cos(2x) & f(0) &= \cos(2 \cdot 0) = 1 \\
f'(x) &= (-2) \cdot \sin(2x) & f'(0) &= (-2) \cdot \sin(2 \cdot 0) = 0 \\
f''(x) &= (-4) \cdot \cos(2x) & f''(0) &= (-4) \cdot \cos(2 \cdot 0) = -4 \\
f'''(x) &= 8 \cdot \sin(2x) & f'''(0) &= 8 \cdot \sin(2 \cdot 0) = 0 \\
f^{(4)}(x) &= 16 \cdot \cos(2x) & f^{(4)}(0) &= 16 \cdot \cos(2 \cdot 0) = 16
\end{aligned}$$

Substituting these values into the formula we get:

$$T_4(x) = 1 + 0 \cdot x + \frac{-4}{2!}x^2 + \frac{0}{3!}x^3 + \frac{16}{4!}x^4 = 1 - 2x^2 + \frac{2}{3}x^4.$$

The Maclaurin polynomial of an even function, i.e. also $f(x) = \cos(2x)$, contains the even powers only, therefore it is an even function:



The function $f : y = \cos(2x)$ and its Maclaurin polynomials of degree $n = 2$ and $n = 4$.

128 – Taylor polynomial

We are going to solve the following problem:

Find a polynomial that approximates the given function f in the neighbourhood of the point $x \in D(f)$ with the smallest possible error.

Theorem (Taylor)

Let f be a function with continuous derivatives up to the order $n+1$ in some neighborhood $N(a)$ of the point $a \in D(f)$. Then the following holds
$$f(x) = T_n(x) + R_{n+1}(x),$$
on $N(a)$, with the remainder term $R_{n+1}(x)$ of the form

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1},$$

where $\xi \in N(a)$.

From the remainder term we can estimate the error caused by taking $T_n(x)$ instead of $f(x)$.

Example

Estimate the error of taking $\mathrm{e} \approx T_5(1) = 2.71\bar{6}$.

Using the Taylor theorem, we compute the value of the remainder:

$$R_6(1) = \frac{f^{(6)}(\xi)}{6!}(x)^6 = \frac{\mathrm{e}^{\xi}}{6!}1^6 \leq \frac{2.72}{6!} \leq 0.0038 = \mathcal{O}(10^{-3}).$$

The first inequality follows from the fact that $\xi \in (0,1)$ and due to monotonicity of $y = \mathrm{e}^x$ we can majorize the error by taking $\xi = 1$, i.e. $f^{(6)}(\xi) = \mathrm{e}^1$. The second inequality results just from rounding up.

The efficiency of the approximation depends heavily on the magnitude of the factors in the remainder term. The error is “small” if

- $(x-a)$ is small, i.e. x is close to a,
- $n!$ is large, i.e. the order n is large,
- $|f^{(n+1)}(x)|$ is numerically small in $N(a)$.

However, the form of the remainder term enables us to compute the error, or at least its order of magnitude.

129 – Taylor polynomial

Example

Estimate the value of $\sin(\frac{1}{2})$ correctly up to 5 decimal places.

We take $y = \sin(x)$ as the function for our approximation. To show the importance of $(x-a)$ being small we compute, for comparison reasons, its Maclaurin polynominal and Taylor polynomial at $a = \frac{\pi}{6}$.
Let us compute the first five derivatives of $\sin(x)$ and their values at $a = 0, \frac{\pi}{6}$. We get:

$$\begin{array}{lll}
f(x) = \sin(x) & f(0) = 0 & f(\frac{\pi}{6}) = \frac{1}{2} \\
f'(x) = \cos(x) & f'(0) = 1 & f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \\
f''(x) = -\sin(x) & f''(0) = 0 & f''(\frac{\pi}{6}) = -\frac{1}{2} \\
f'''(x) = -\cos(x) & f'''(0) = -1 & f'''(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2} \\
f^{(4)}(x) = \sin(x) & f^{(4)}(0) = 0 & f^{(4)}(\frac{\pi}{6}) = \frac{1}{2} \\
f^{(5)}(x) = \cos(x) & f^{(5)}(0) = 1 & f^{(5)}(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}
\end{array}$$

Substituting these values into the Maclaurin and Taylor formula we get:

$$\begin{aligned}
M_5(x) &= 0 + 1 \cdot x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 \\
&= x - \frac{1}{6}x^3 + \frac{1}{120}x^5. \\
T_5(x) &= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + \\
&\quad + \frac{1}{48}\left(x - \frac{\pi}{6}\right)^4 + \frac{\sqrt{3}}{240}\left(x - \frac{\pi}{6}\right)^5.
\end{aligned}$$

Let us start with the Maclaurin expansion. Its value at $x = \frac{1}{2}$ is

$$T_5\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{48} + \frac{1}{3840} = \frac{1841}{3840} = 0.4794\ldots$$

and the error, estimated using the formula for the remainder,

$$R_6\left(\frac{1}{2}\right) = \frac{|-\sin(\xi)|}{(6)!}\left(\frac{1}{2}\right)^6 \le \frac{|\sin\frac{\pi}{6}|}{720} \cdot \frac{1}{64} \doteq 0.00001085\ldots = \mathcal{O}(10^{-5}).$$

Hence, we have achieved the desired precision. Note that the inequality follows by an argument analogous to the previous example. For $\frac{\pi}{6} > \frac{1}{2}$ the value $\sin\frac{\pi}{6} > \sin(\xi)$ for any $\xi \in (0, \frac{1}{2})$ due to the monotonicity of $\sin(x)$ on the interval $(0, \frac{\pi}{6})$.

Before we proceed with the Taylor formula at $x = \frac{1}{2}$, let us note that we need the values of π and $\sqrt{3}$ (at least with the required precision) in order to use the Taylor formula. Anyway, we just estimate the errors from the remainder:

$$\begin{aligned}
R_4\left(\frac{1}{2}\right) &= \frac{|-\sin(\xi)|}{(4)!}\left(\frac{1}{2} - \frac{\pi}{6}\right)^4 \le \frac{\frac{1}{2}}{48} \cdot \left(\frac{3-\pi}{6}\right)^4 = \mathcal{O}(10^{-6}), \\
R_6\left(\frac{1}{2}\right) &= \frac{|-\sin(\xi)|}{(6)!}\left(\frac{1}{2} - \frac{\pi}{6}\right)^6 \le \frac{\frac{1}{2}}{720} \cdot \left(\frac{3-\pi}{6}\right)^6 = \mathcal{O}(10^{-9}).
\end{aligned}$$

It is clear that due to the fact that $(x-a)$ is smaller the required precision is obtained much more quickly.

130 – Taylor polynomial

Example

Estimate the value of $\ln(2)$ correctly up to 3 decimal places.

We take $y = \ln(x)$ as the function for our approximation and compute its Taylor polynomial at $a = 1$.
Let us compute the first five derivatives of $\ln(x)$ and their values at $a = 1$. We get:

$$
\begin{array}{ll}
f(x) = \ln(x) & f(1) = 0 \\
f'(x) = \dfrac{1}{x} & f'(1) = 1 \\
f''(x) = -\dfrac{1}{x^2} & f''(1) = -1 \\
f'''(x) = \dfrac{2}{x^3} & f'''(1) = 2 \\
f^{(4)}(x) = -\dfrac{6}{x^4} & f^{(4)}(1) = -6 \\
f^{(5)}(x) = \dfrac{24}{x^5} & f^{(5)}(1) = 24
\end{array}
$$

Substituting these values into the Taylor formula we get:

$$
\begin{aligned}
T_5(x) &= 0 + 1 \cdot (x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 + \frac{24}{5!}(x-1)^5 \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5
\end{aligned}
$$

and its value for $x = 2$ is

$$T_5(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{47}{60} = 0.78\bar{3},$$

which differs from the correct value $\ln(2) \approx 0.6931$ by 0.09. It is easy to see from the Taylor expansion that the contribution of the n-th term is $\pm\frac{1}{n}$. Hence, the result will be correct up to three decimal places only for $n = 1000$. This result should make us aware of the limits of applicability of the Taylor theorem.

131 – Taylor polynomial

Exercise

Write down the Maclaurin polynomial of the order 4 for the function

a) $f : y = x^2 \mathrm{e}^x$,

b) $g : y = \mathrm{e}^x \cdot \cos(x)$.

Hints

Maclaurin polynomial is the Taylor polynomial at the point $a = 0$:

$$M_4(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

132 – Taylor polynomial

Exercise

Estimate the values of

a) $\sin(1^\circ)$

b) $\sin(1)$

c) $\tan(1)$

d) $\arctan(1)$

correctly up to 3 decimal places.

Hints

Taylor polynomial of the order n at the point a:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Error:

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

133 – Local extremes

Which x makes $f(x)$ as large as possible? Where is the smallest $f(x)$? Without calculus, we would have been forced to compute the values of $f(x)$ and compare them.

Definition

We say that the function f has a **local maximum** at the point $x_0 \in D(f)$ if there exists a punctured neighborhood $N(x_0)$ of the point x_0 such that

$$f(x) < f(x_0) \qquad \text{for all } x \in N(x_0).$$

Similarly, the function f has a **local minimum** at the point $x_0 \in D(f)$ if

$$f(x) > f(x_0) \qquad \text{for all } x \in N(x_0)$$

The results would be unsatisfactory. Thanks to calculus, we can obtain the necessary information from df/dx.

How do you identify maximum or minimum?
Typically, the slope is zero. *If df/dx exists, it must be zero.* The tangent line is horizontal. The graph changes from increasing to decreasing. The slope changes from positive to negative. This turning point of f' is called **a stationary point**.

It is also possible that the graph has a corner, and thus no derivative. These points are called **rough points**.

Last but not least we should check the **endpoints** of the domain.

We summarize the situation as follows:

Theorem (Fermat's theorem)

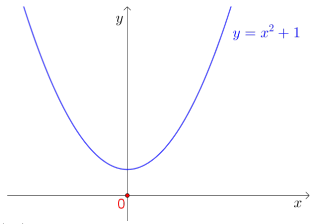
If f is differentiable at $x_0 \in (a,b)$, and $f'(x_0) \neq 0$, then x_0 is not a local extremum of f.

Fermat's theorem gives us a necessary condition for the existence of a local extremum. As a contrapositive statement, it allows us to rule out the points, where there is no extremum. The remaining points are co called **critical points**, suspected of the existence of an extremum. They are of the following three types:

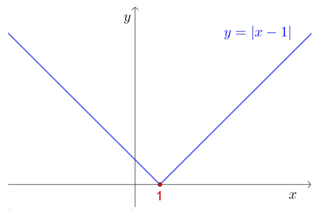
a) **stationary points**, where $df/dx = 0$,

b) **rough points**, where df/dx does not exist,

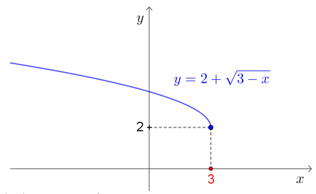
c) **endpoints** of the domain.



(a) Stationary point $x = 0$ of the function $y = x^2 + 1$.



(b) Rough point $x = 1$ of the function $y = |x - 1|$.



(c) Endpoint $x = 3$ of the function $y = 2 + \sqrt{3 - x}$.

For the most common case of the stationary points there is an easy-to-check sufficient condition for the existence of a local extremum.

Theorem (Second derivative test)

If $f'(x_0) = 0$ and $f''(x_0) > 0$, then the function f has a local minimum at the point x_0.

If $f'(x_0) = 0$ and $f''(x_0) < 0$, then the function f has a local maximum at the point x_0.

134 – Minimum and maximum problems

To find a maximum or minimum, we just find critical points of f. We solve the equation $f'(x) = 0$ and then check the rough points, where $f'(x)$ doesn't exist, and endpoints. The idea is clear, but to be honest, that is not where the problem starts.

In reality, the first (and often the hardest) step is to choose the unknown variable, which should be minimized, and find the function describing its behaviour. We and only we ourselves decide what will be x and how would $f(x)$ look like. The equation $df/dx = 0$ comes out by a standard procedure, often easily with the help of computer. On the other hand no computer so far is able to analyse the situation and propose the correct form of f.

The heart of this subject is in word problems. The procedure of solving the problem can be divided into steps:

1. Find (propose) the quantity x to be minimized or maximized.
2. Express the quantity x as a function $f(x)$.
3. Compute $f'(x)$ and solve $f'(x) = 0$.
4. Check all critical points for f_{min} and f_{max}.

135 – Minimum and maximum problems

Example

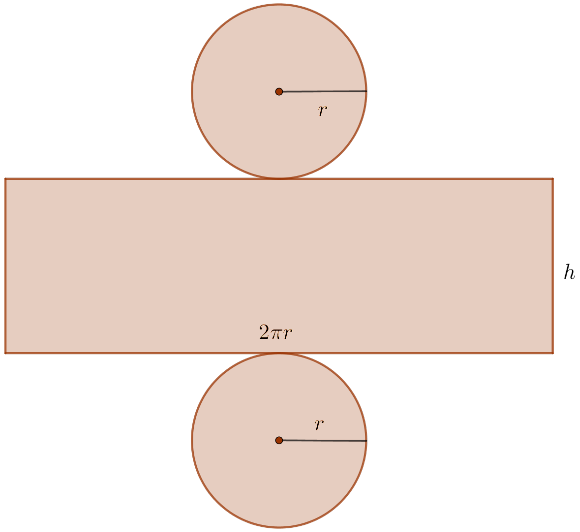
Barrel problem.
The army is looking for a big amount of 500 litre barrels for fuel. Due to the shortage of the metal plates, you should propose a shape of the barrels (the radius r and the height h) spending the least of the worthy material.

The volume of the barrel is obtained from the formula for the cylinder

$$V = \pi r^2 h = 500\,\mathrm{dm}^3.$$

The surface of the cylindrical barrel consists of two circles (the bottom and the top) and the rectangular body. Its area A is their sum:

$$A = 2(\pi r^2) + 2\pi r h.$$



Surface area of the barrel consisting of two circles and a rectangle

The function A should be made minimal. However, it depends on the two unknowns r and h and we need to minimize function of one variable only. The two variables are connected through the formula for V, which gives us the possibility of expressing the height h in terms of r. Indeed,

$$h = \frac{500}{\pi r^2}.$$

In this way we obtain the formula for A as a function of one variable r:

$$A(r) = 2(\pi r^2) + 2\pi r \frac{500}{\pi r^2} = 2(\pi r^2) + \frac{1000}{r},$$

with the domain $D(A) = (0, \infty)$.
The rest is pretty standard. To find the minimum of the function $A(r)$ we first compute the derivative

$$\frac{dA}{dr} = 4\pi r + 1000 \cdot \frac{-1}{r^2}$$

and use the Fermat theorem $dA/dr = 0$ to compute the stationary points. This equation has unique solution

$$r_0 = \sqrt[3]{\frac{1000}{4\pi}} \doteq 4.301\,\mathrm{dm}.$$

We can check that this is the minimum using the second derivative test. Indeed,

$$\frac{d^2A}{dr^2} = 4\pi + \frac{2000}{r^3} > 0.$$

Remark

Notice that $h = 2r$, which means that height is equal to the diameter of the barrel. This is another manifestation of a symmetry so often seen in the minimization problems.

136 – Minimum and maximum problems

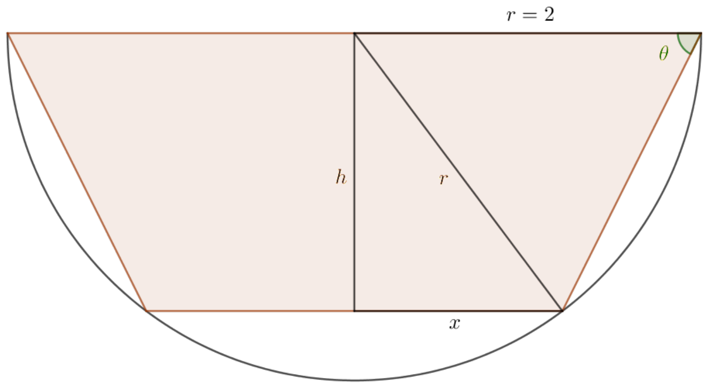
Exercise

Drainage canal.
The company building a drainage canals should dredge a semicircle thalwegs of radius 2 m. The canals should be concreted into the form of a trapezoid with the bottom parallel to the surface, see figure below. Find the shape of the trapezoid so that is allows maximal possible streaming (in that case the trapezoid has maximal area).

Hints

Express the area of the trapezoid in terms of θ.

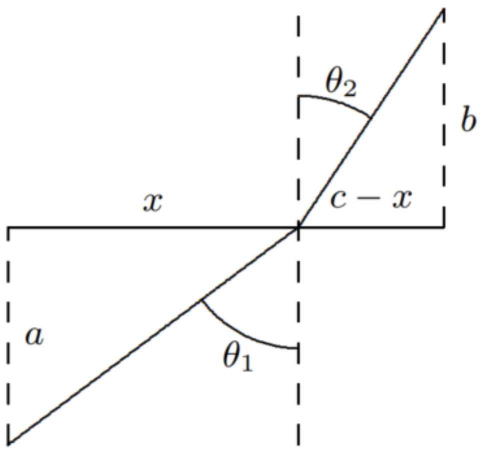
Sectional view of the drainage canal is as follows:



137 – Minimum and maximum problems

Exercise

Baywatch.
You are standing near the side of a large wading pool of uniform depth when you see a child in trouble. You can run at a speed $v_1 = 7.1\,\mathrm{m/s}$ on land and swim at the speed $v_2 = 1.6\,\mathrm{m/s}$ in the water. Your perpendicular distance from the side of the pool is a, the child's perpendicular distance is b, and the distance along the side of the pool between the closest point to you and the closest point to the child is c (see the figure below). Without stopping to do any calculus, you instinctively choose the quickest route (shown in the figure) and save the child. Our purpose is to derive a relation between the angle θ_1 your path makes with the perpendicular to the side of the pool when you're on land, and the angle θ_2 your path makes with the perpendicular when you're in the water. To do this, let x be the distance between the closest point to you at the side of the pool and the point where you enter the water. Write the total time you run (on land and in the water) in terms of x and find its minimum.



Hints

The result is called "Snell's law" or the "law of refraction".

138 – Minimum and maximum problems

Exercise

Fencing a pasturage.
A rancher needs to fence a rectangular pasturage area next to a straight river, using 1200 m of fencing. The side next to the river will not be fenced, to allow the cattle drinking and freshening up in the river. Advice the rancher the dimensions of the rectagnle to maximize the area of the pasturage. What is the maximum area?

Exercise

Running a hotel.
A new 120-room hotel to be opened in Prague is setting up its prices. The manager knows that they will rent all of its rooms if they charge €50 per room and for each €2 increase per room, three fewer rooms will be rented per night. What rent per room would maximize the profit per night?

139 – Minimum and maximum problems

Exercise

Cutting a beam.
The strength of a rectangular beam is given by $S = \nu \cdot w \cdot d^2$, with width w and depth d. Find the dimensions of the strongest beam that can be cut from a cylindrical log of larch wood ($\nu = 0.35$) of radius $r = 30\,\text{cm}$.

Exercise

Shipping a parcel to the USA.
The U.S. post office will accept a box for shipment only if the sum of the length and girth (distance around) is at most 274 cm. Find the dimensions of the largest acceptable box with square front and back.

140 – Monotonicity

Suppose that df/dx is positive at a point x_0. Then the tangent line slopes upward. Therefore it is increasing (as a linear function) and the function $f(x)$ itself is also increasing at the point x_0.

Theorem

If $f'(x_0) > 0$, then the function f is strictly increasing at the point x_0.

If $f'(x_0) < 0$, then the function f is strictly decreasing at the point x_0.

This “local” theorem describing behaviour can be easily generalized to “global” open intervals:

Theorem

If $f'(x_0) > 0$ for all $x \in I = (a, b)$, then the function f is strictly increasing on the interval I.

If $f'(x_0) < 0$ for all $x \in I = (a, b)$, then the function f is strictly decreasing on the interval I.

Remark

Note that the preceding theorem cannot be generalized on the union of intervals.

141 – Monotonicity

Example

Find the intervals of the strict monotonicity of the function

$$y = x^2 - 12\ln(x-1).$$

We start with computing the domain. Here, we have only one condition, required by the definition of the logarithmic function, namely $x - 1 > 0$. Therefore, the domain is $D = (1, \infty)$.
We proceed with the first derivative,

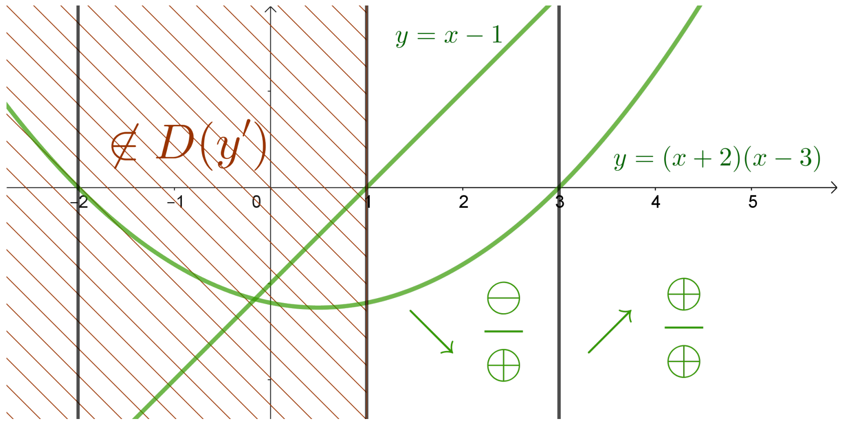
$$y' = 2x - 12 \cdot \frac{1}{x-1}.$$

Note that $D_{y'} = D = (1, \infty)$, even if the function $2x - 12/(x-1)$ has larger domain.
As we need to solve two inequalities, $y' > 0$ and $y' < 0$, we find the zero points of y'.

$$\begin{aligned} y' &= 2x - 12 \cdot \frac{1}{x-1} = \frac{2x(x-1) - 12}{x-1} = \frac{2x^2 - 2x - 12}{x-1} \\ &= \frac{2(x+2)(x-3)}{x-1} = 0 \end{aligned}$$

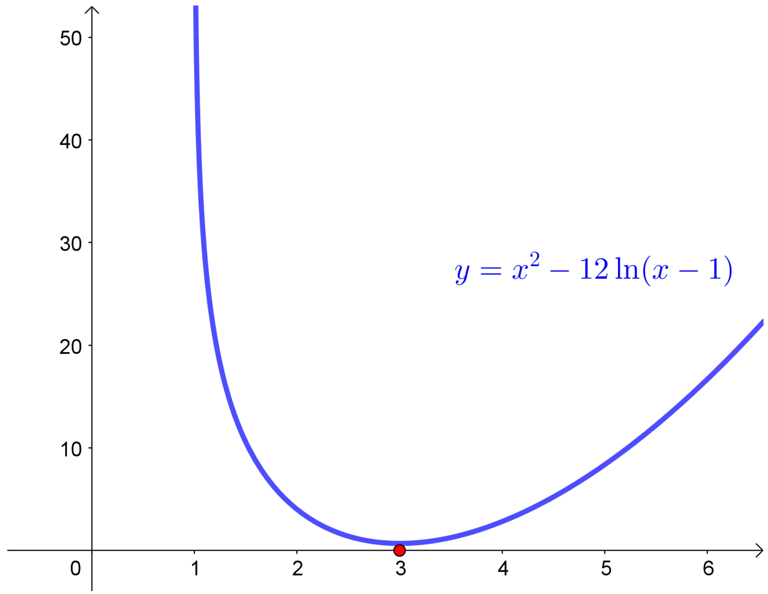
This rational expression has three roots, $x_i = -2, 1, 3$, which divide the domain to the subintervals, where y' does not change its sign and therefore it will be possible to decide which inequality is fulfilled on a particular interval.
As the roots $x_1 = -2$, $x_2 = 1$ do not belong to the domain, we have just single root $x = 3$ and we solve the inequalities by the graphical method on the intervals $(1, 3)$ and $(3, \infty)$.

The denominator is positive in both cases, so the sign of the numerator decides on the result:



Intervals of monotonicity.

The function is increasing on the interval $(3, \infty)$ and decreasing on $(1, 3)$.



Intervals of monotonicity.

142 – Monotonicity and local extremes

Example

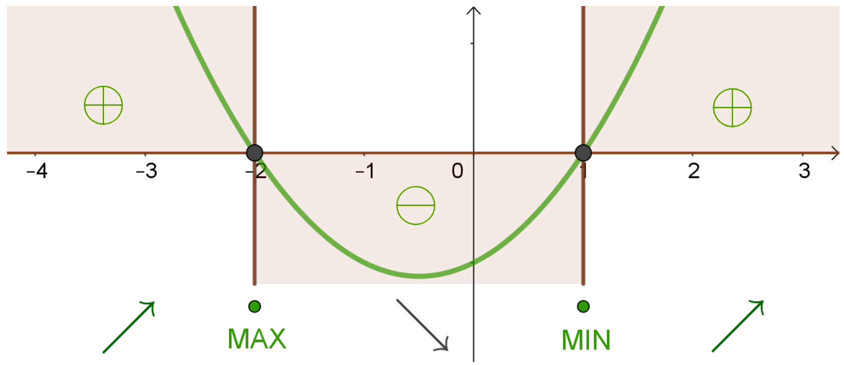
Find intervals of monotonicity and all local extremes of the function

$$y = 2x^3 + 3x^2 - 12x + 24.$$

We start with checking the domain, here it is easy, $D = \mathbb{R}$. We proceed with the first derivative, $y' = 6x^2 + 6x - 12$.
We solve both tasks at once. As we need to solve the the equation $y' = 0$ and the two inequalities, $y' > 0$ and $y' < 0$, we first find the zero points of y', which divide the domain to the subintervals, where y' does not change its sign and therefore it is easy to decide which inequality is fulfilled on a particular interval.
Factoring out the common multiple 6 we rewrite the equation $y' = 0$ to the form

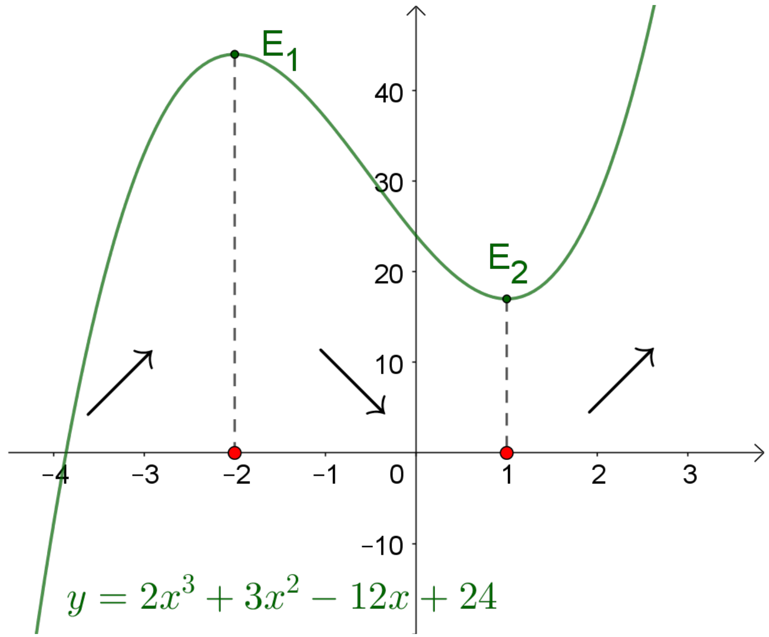
$$\begin{aligned} 6(x^2 + x - 2) &= 0 \\ 6(x+2)(x-1) &= 0 \end{aligned}$$

So the stationary points are $x_1 = -2$, $x_2 = 1$, we have no rough points and no endpoints. We easily solve the inequalities by the graphical method:



Intervals of monotonicity.

The function is increasing on the intervals $(-\infty, -2)$ and $(1, \infty)$ (however not on their union!) and decreasing on $(-2, 1)$.
Therefore, there is a local maximum at $x = -2$ and a local minimum at $x = 1$. Note that we don't even need the second derivative test in this case.



Graph of the function with marked intervals of monotonicity and local extremes.

143 – Monotonicity and local extremes

Exercise

For the given functions:

a) $y = x + \ln\left(2x^2 - x + 1\right)$ b) $y = \ln\sqrt[3]{x^2} + x$ c) $y = \dfrac{\ln x}{1 - \ln x}$

find the intervals, where the function is increasing (respectively decreasing). Compute the coordinates of the local maxima and minima.

Hints

Algorithm:

1. Find $D(f)$.
2. Compute $f'(x)$.
3. Find stationary points
4. Determine intervals with $f'(x) > 0$ and $f'(x) < 0$.
5. Decide about local extremes.

144 – Monotonicity and local extremes

Exercise

For the given functions:

a) $y = \ln\left(\dfrac{1-x}{x+2}\right)$

b) $y = \dfrac{1}{x \cdot \ln x}$

c) $y = \ln\left(x^2\right) - x^2$

find the intervals, where the function is increasing (respectively decreasing). Compute the coordinates of the local maxima and minima.

Hints

Algorithm:

1. Find $D(f)$.
2. Compute $f'(x)$.
3. Find stationary points
4. Determine intervals with $f'(x) > 0$ and $f'(x) < 0$.
5. Decide about local extremes.

145 – Monotonicity and local extremes

Exercise

For the given functions:

a) $y = \sqrt{\frac{x-2}{3-x}}$

b) $y = \arctan\left(x + \frac{1}{x}\right)$

c) $y = \mathrm{e}^{\cos(2x)}$

d) $y = 1 + 2\sin^3 x$

find the intervals, where the function is increasing (respectively decreasing). Compute the coordinates of the local maxima and minima.

Hints

Algorithm:

1. Find $D(f)$.
2. Compute $f'(x)$.
3. Find stationary points
4. Determine intervals with $f'(x) > 0$ and $f'(x) < 0$.
5. Decide about local extremes.

146 – Monotonicity and local extremes

Exercise

For the given functions:

a) $y = (1 + 2\sin x)^3$

b) $y = \sqrt{(x-1)\cdot(x+1)\cdot(x+3)}$

c) $y = \dfrac{x^3}{x^2 + 4x + 3}$

d) $y = \dfrac{x^2 + 4}{x^2 - 3x + 4}$

find the intervals, where the function is increasing (respectively decreasing). Compute the coordinates of the local maxima and minima.

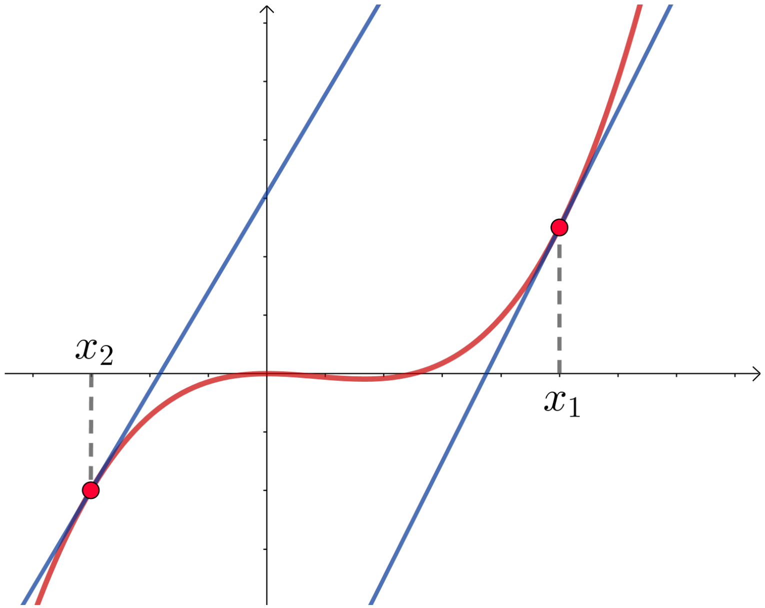
Hints

Algorithm:

1. Find $D(f)$.
2. Compute $f'(x)$.
3. Find stationary points
4. Determine intervals with $f'(x) > 0$ and $f'(x) < 0$.
5. Decide about local extremes.

147 – Convexity as an expression of a curvature of the graph

The curvature of the graph can be also described by the tangent line, hence the derivative.
If the *tangent line* to the graph of the function $y = f(x)$ at the point $[x_0, f(x_0)]$ *is lying below* the graph of the function at some neigborhood of x_0, we call the function **convex** at the point (x_0). Similarly, if the tangent line is lying above the graph, it is **concave** at the point (x_0).



A function convex at x_1 and concave in x_2.

These considerations are not convenient for practical calculations, therefore we have the following easy criterion.

Theorem

If $f''(x_0) > 0$, then the function f is convex at the point x_0.

If $f''(x_0) < 0$, then the function f is concave (convex negative) at the point x_0.

Does the theorem extend to open intervals? Yes and very easily:

Theorem

If $f''(x_0) > 0$ for all $x \in (a, b)$, then the function f is convex on the interval (a, b).

If $f''(x_0) < 0$ for all $x \in (a, b)$, then the function f is concave on the interval (a, b).

The latter theorem

1. geometrically justifies our second derivative test,
2. cannot be generalized on the union of intervals.

148 – Convexity

Example

Find the intervals of the convexity for the function

$$y = x^4 - 4x^3.$$

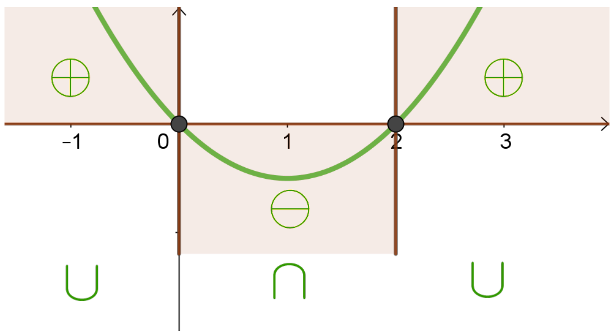
We start with checking the domain, $D = \mathbb{R}$.
We proceed with the first and second derivative,

$$\begin{aligned} y' &= 4x^3 - 12x^2, \\ y'' &= 12x^2 - 24x \end{aligned}$$

Similarly as by the monotonicity, we need to solve two inequalities, only here with the second derivatives, $y'' > 0$ and $y'' < 0$.
Again, we first find the zero points of y''. Factoring out the common multiple 12 we rewrite the equation $y'' = 0$ to the form

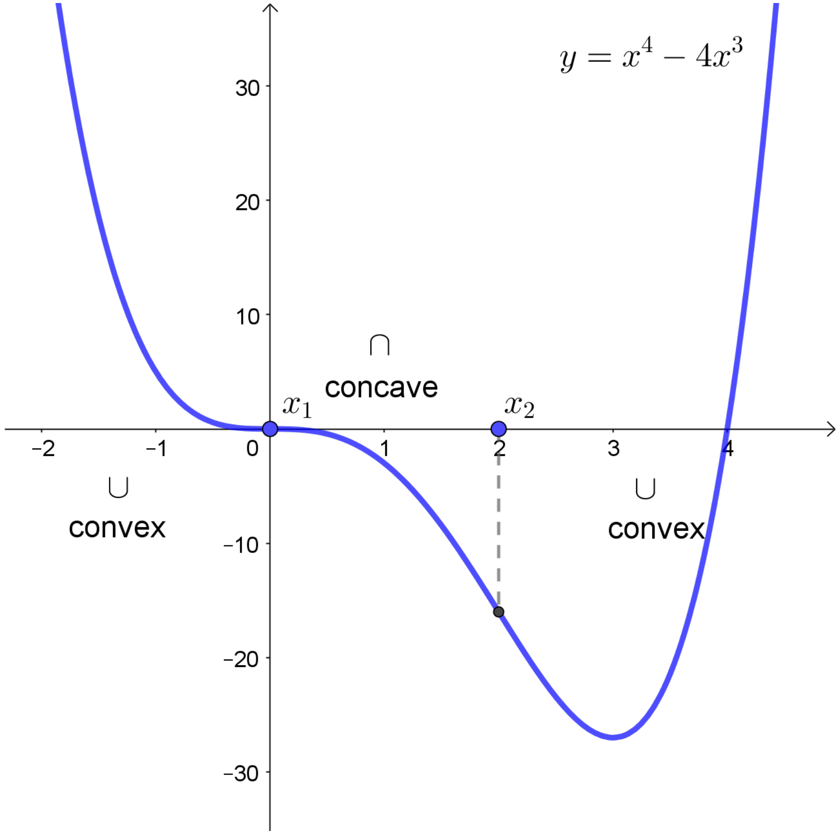
$$\begin{aligned} 12(x^2 - 2x) &= 0 \\ 12x(x-2) &= 0 \end{aligned}$$

So the stationary points are $x_1 = 0$, $x_2 = 2$, with no rough points and no endpoints. Again, We easily solve the inequalities by the graphical method:



Intervals of convexity and concavity. The symbol $\oplus$ denotes the interval, where $y'' > 0$, the symbol $\ominus$ denotes $y'' < 0$.

Notice that the point $x = 0$ is not an extremum, even though it holds $f'(0) = 0$.



The function $y = x^4 - 4x^3$, its intervals of convexity and concavity.

149 – Inflection points

We describe the points on a graph, where the curvature changes of sign. In particular, it is a point where the function changes from being concave to convex, or vice versa.

Definition

We say that the point $x_0 \in D$ is the **inflection point** of the function f, if there exists a neighborhood $N_\delta(x_0)$ of the point x_0 such that

$$f''(x-\epsilon) \cdot f''(x+\epsilon) < 0 \qquad \text{for all } \epsilon \in (0, \delta).$$

This is not practical criterion that can be efficiently used by the computation. We state such a criteria in the following theorems.

Theorem (necessary condition for the existence of an inflection point)

If the function f has an inflection point x_0, then

$$f''(x_0) = 0.$$

Theorem (sufficient condition for the existence of an inflection point)

If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then the function f has an inflection point x_0.

The thoerem can be further precised. At the inflection point x_0 the lowest non-zero derivative is of an odd order.

Example

Find all inflection points of the function $y = \sin(2x)$.

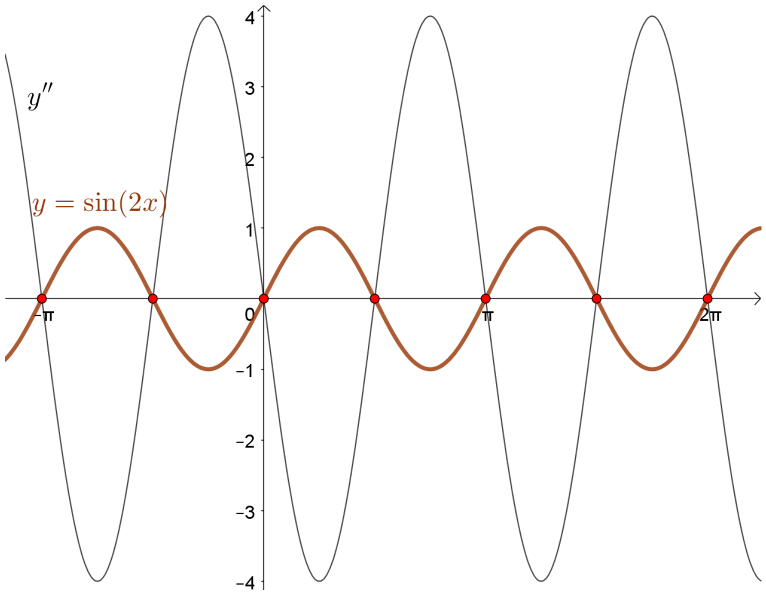
We compute the second derivative y''

$$\begin{aligned} y' &= \cos(2x) \cdot 2, \\ y'' &= 2(-1)\sin(2x) \cdot 2 = (-4)\sin(2x) \end{aligned}$$

and set $y'' = 0$. We see that the inflection points coincide with the intersection point of y with the x axis,

$$x_i = \left\{ 0 + k\frac{\pi}{2} \,\middle|\, k \in \mathbb{Z} \right\}.$$

Moreover, if $y = \sin(2x)$ is positive, then it is concave, and if y is negative, it is convex.



The intervals of convexity and concavity of the function $y = \sin(2x)$.

150 – Inflection points

Example

Find all inflection points of the function $y = (x^2 - 1)^5$.

We compute the second derivative y''

$$\begin{aligned} y' &= 5(x^2-1)^4 \cdot 2x = 10x(x^2-1)^4, \\ y'' &= 10(x^2-1)^4 + 10x(x^2-1)^3 \cdot 2x = 10(x^2-1)^3(3x^2-1) \end{aligned}$$

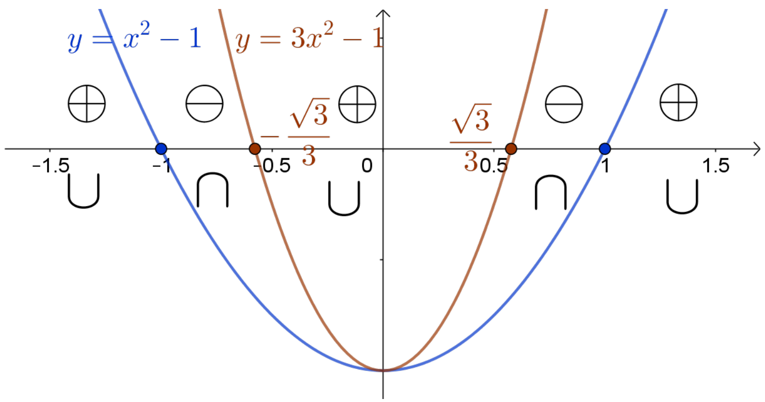
and set $y'' = 0$. It can be decomposed on two equations,

$$x^2 - 1 = 0 \qquad \text{or} \qquad 3x^2 - 1 = 0$$

with the roots

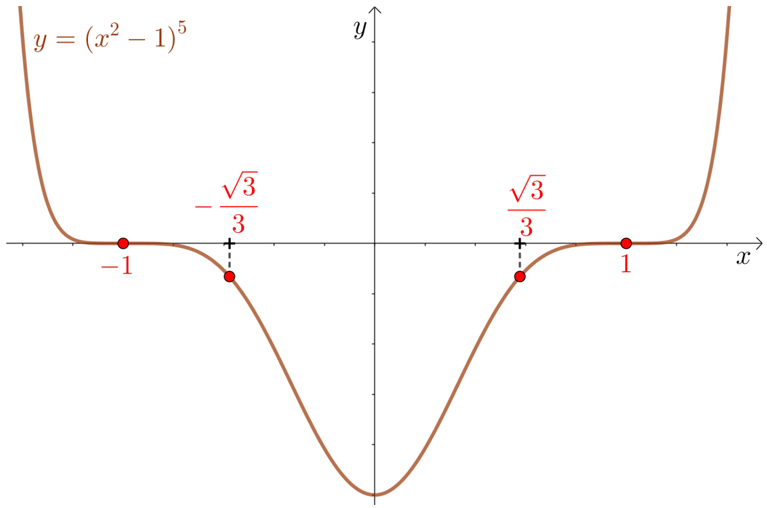
$$x_{1,2} = \pm 1, \qquad x_{3,4} = \pm\frac{\sqrt{3}}{3}.$$

Computing the values of y''' at all the critical points would be lengthy, so we decide on the intervals of convexity and utilize the definition.



The intervals of convexity and concavity of the function $y = (x^2 - 1)^5$.

Hence, in all four points $x_1, \ldots, x_4$, the socnde derivative changes its sign, they are all inflection points. We got this from the intervals of convexity (concavity respectively) for free.



The function $y = (x^2 - 1)^5$ and its inflection points determined from the intervals of convexity and concavity.

151 – Curvature and inflection points

Exercise

For the given functions:

a) $y = \ln(x) - \sqrt{x}$

b) $y = \ln\left(x^2 - 1\right)$

c) $y = \ln\left(\frac{1}{\sqrt{x}}\right) - x^2$

find the point of inflection. Next, determine the intervals, where the function is convex (respectively concave).

Hints

Algorithm:

1. Find $D(f)$.
2. Compute $f''(x)$.
3. Find inflection points
4. Determine intervals with $f''(x) > 0$ and $f''(x) < 0$.

152 – Curvature and inflection points

Exercise

For the given functions:

a) $y = \ln(x-1) + \dfrac{x^2}{2}$ b) $y = \left(1 - \dfrac{1}{x}\right)^3$ c) $y = \sqrt{x^2 - 1}$

find the point of inflection. Next, determine the intervals, where the function is convex (respectively concave).

Hints

Algorithm:

1. Find $D(f)$.
2. Compute $f''(x)$.
3. Find inflection points
4. Determine intervals with $f''(x) > 0$ and $f''(x) < 0$.

153 – Curvature and inflection points

Exercise

For the given functions:

a) $y = \dfrac{\mathrm{e}^x}{x}$

b) $y = x^4 \cdot \mathrm{e}^x$

c) $y = \sqrt{\mathrm{e}^{1-x}}$

d) $y = \mathrm{e}^{x^2-1}$

find the point of inflection. Next, determine the intervals, where the function is convex (respectively concave).

Hints

Algorithm:

1. Find $D(f)$.
2. Compute $f''(x)$.
3. Find inflection points
4. Determine intervals with $f''(x) > 0$ and $f''(x) < 0$.

154 – Curvature and inflection points

Exercise

For the given functions:

a) $y = \dfrac{2 - x^2}{e^x}$

b) $y = (x^2 - 2) \cdot e^{x-1}$

c) $y = \sin(x) \cdot e^x$

d) $y = e^{\frac{2}{1-x}}$

find the point of inflection. Next, determine the intervals, where the function is convex (respectively concave).

Hints

Algorithm:

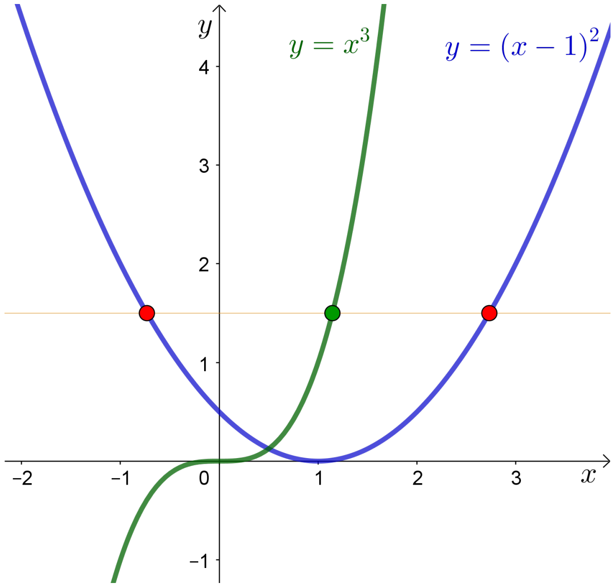
1. Find $D(f)$.
2. Compute $f''(x)$.
3. Find inflection points
4. Determine intervals with $f''(x) > 0$ and $f''(x) < 0$.

155 – Transposing formulae

The formula $y = f(x)$ describes evolution of physical quantity f depending on another physical quantity x. In physics, some processes are idealised as *reversible*. In this case, the function describing the value quantity f is uniquely determined by the value of x. We say that f is **one-to-one**. This property can be formulated geometrically as follows:

Definition

The function $y = f(x)$ is called **one-to-one** if and only if the graph of f and a horizontal line $y = q$ have at most one intersection point for any $q \in \mathbb{R}$.



The prototypes of one-to one functions are odd powers, e.g. $y = x^3$. On the other hand, the even powers are not one-to-one.

There is an easy-to-check criterion for any f to be one-to-one.

Theorem

- If the function f is increasing, it is one-to-one.
- If the function f is decreasing, it is one-to-one.

156 – Transposing formulae

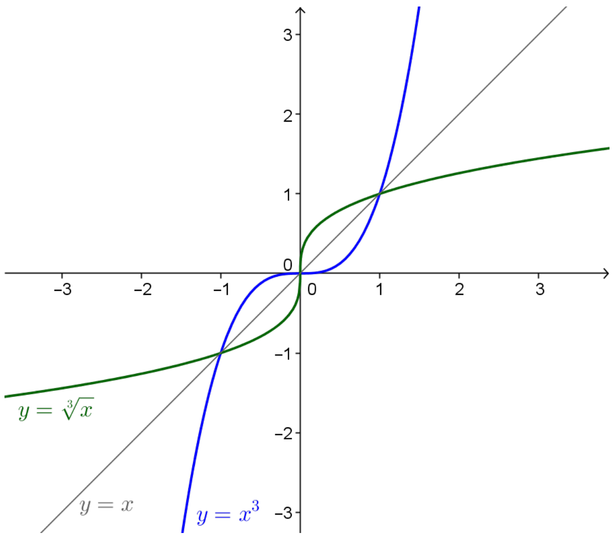
The function describing the process returning the system into the original state can be computed by transposing the formula for f. The resulting function is called **the inverse function to** f and denoted by f^{-1}.

Theorem

For the one-to-one function $y = f(x)$ with the domain D_f and range I_f, there exists unique one-to-one inverse function to f defined on $I_f = D_{f^{-1}}$ by the formula

$$f^{-1}(y) = x \qquad \text{if and only if} \qquad f(x) = y.$$

Moreover, if f^{-1} is the inverse function to f, then f is the inverse function to f^{-1}; i.e. the inverse relation is mutual. Therefore, the graphs of the mutually inverse functions are axially symmetric with respect to the line (axis) $y = x$.



The graphs of the mutually symmetric functions are axially symmetric.

Theorem

- If f is increasing then f^{-1} is increasing.
- If f is decreasing then f^{-1} is decreasing.

Remark

Note that the notation of the inverse function is ambiguous. In this context, the superscript *does not* mean "f to the power of -1", as the inverse is made with respect to the composition of functions and not multiplication. Therefore, it holds

$$f^{-1}\left(f(x)\right) = x$$

and *not* $f^{-1}(x) \cdot f(x) = 1$.
In another words, the inverse function f^{-1} is *not* equal to the reciprocal function $y = 1/f$, i.e.

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

157 – Transposing formulae

What is the relation between the domain and range of the mutually inverse functions?
According to the definition and properties of f^{-1} it holds:

- $D_f = I_{f^{-1}}$ and $I_f = D_{f^{-1}}$,
- for every $x \in D_f$ and $y \in D_{f^{-1}}$ it holds

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(y)) = y.$$

We often need to transpose a formula, which is not one-to-one, e.g. for the function $y = x^2$. We now describe how this could be done.

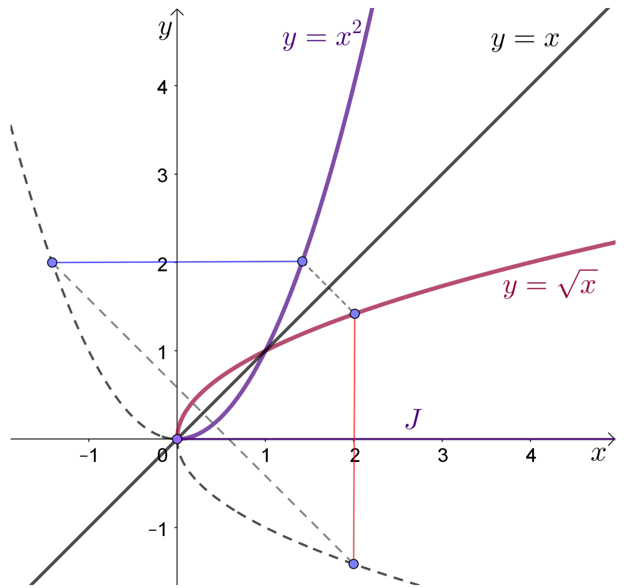
The procedure of finding the inverse function $y = f^{-1}(x)$ **to the function** $y = f(x)$ can be described as follows:

1. find the domain D_f.
2. Decide if the function f is one-to-one. If it is not, find the biggest interval J, where f is one-to-one, and take J as the domain of f, make a **restriction** of f on J. It is denoted by $f\mid_J$.
3. Compute the transposed formula for its inverse function.
4. Compute the domain and the range of the inverse function.

Example

Construct an inverse function to the function $f : y = x^2$.

The function f is even, hence not one-to one. We choose $J = \langle 0, \infty)$ and construct f^{-1} for $f\mid_J$. As on a larger interval f is not one-to-one, the resulting f^{-1} would not be a function.



Constructing an inverse function f^{-1} to the even function, $y = x^2$.

Notice that there might be a freedom of choice in the decision about the interval J. In the preceding example, we could choose $J = (-\infty, 0\rangle$. In practise, we make the decision based on which points $x \in D_f$ we want to map.

158 – Transposing formulae

Example

Let be given the function

$$f(x) : y = 1 - \ln\left(-1 + \sqrt{x}\right).$$

Decide if f is one-to-one. Compute the transposed formula for its inverse function f^{-1}. Write down the domains D_f, $D_{f^{-1}}$ and ranges I_f, $I_{f^{-1}}$.

We start with the domain of f. From the two conditions involved by the square root, $x \geq 0$, and the logarithm, $-1 + \sqrt{x} > 0$, we get $D_f = (1, \infty)$. Next we check if the function f is one-to-one, based on its monotonicity. We compute f',

$$y' = -\frac{1}{\sqrt{x} - 1} \cdot \frac{1}{2\sqrt{x}}.$$

This expression is negative on D_f, as the first fraction is positive for $x > 1$, i.e. exactly on D_f, and the second fraction is positive even on the bigger interval $\langle 0, \infty)$. Therefore, f is decreasing, thus one-to-one.
We compute the formula for the inverse by switching $x \leftrightarrow y$ and expressing y:

$$\begin{aligned}
x &= 1 - \ln\left(-1 + \sqrt{y}\right) \\
\ln\left(-1 + \sqrt{y}\right) &= 1 - x \\
-1 + \sqrt{y} &= \mathrm{e}^{1-x} \\
\sqrt{y} &= 1 + \mathrm{e}^{1-x} \\
y &= \left(1 + \mathrm{e}^{1-x}\right)^2
\end{aligned}$$

The domain $D_{f^{-1}} = \mathbb{R} = I_f$ and the range $I_{f^{-1}} = (1, \infty) = D_f$, as expected.

159 – Transposing formulae

Example

Let be given the function

$$f(x) : y = 3 - \frac{2}{1 + 2x + x^2}.$$

Decide if f is one-to-one. If it is not, find the biggest interval J, where f *is* one-to-one and take $f\,|_J$.
Next, compute the transposed formula for its inverse function f^{-1}. Write down the domains D_f, $D_{f^{-1}}$ and ranges I_f, $I_{f^{-1}}$.

We first rewrite the formula as follows:

$$f : y = 3 - \frac{2}{(1 + x)^2}.$$

Then, we see the domain $D = \mathbb{R} \setminus \{-1\} = (-\infty, -1) \cup (-1, \infty)$ more easily.
In order to show that f is one to one, we show the intervals of monotonicity. The derivative

$$y' = (-2)(-2)(1 + x)^{-3} = \frac{4}{(1 + x)^3}$$

is positive on $J_1 = (-1, \infty)$ and negative on $J_2 = (-\infty, -1)$. Therefore, f is not monotone. However, if we choose just one of the two intervals, f would be monotone, therefore one-to-one. It can be also checked by the graph.
We choose J_1 and restrict f on this interval, i.e. take J_1 as the domain of f. Therefore, f is increasing on J_1, hence one-to-one.

We can also compute the inverse. We do it by switching $x \leftrightarrow y$ in the formula for f and expressing y:

$$\begin{aligned} x &= 3 - \frac{2}{(1 + y)^2} \\ \frac{2}{(1 + y)^2} &= 3 - x \\ (1 + y)^2 &= \frac{2}{3 - x} \\ 1 + y &= \sqrt{\frac{2}{3 - x}} \\ y &= \sqrt{\frac{2}{3 - x}} - 1 \end{aligned}$$

Just note that we have taken the *positive* square root in the next-to-last line, in order that $x \stackrel{f}{\to} y \stackrel{f^{-1}}{\to} x$. Therefore, we get the range $I_{f^{-1}} = (-1, \infty) = J_1$, i.e., the interval we chose as D_f.
The domain, $D_{f^{-1}} = (-\infty, 3)$, coincides with the range I_f.

If we take J_2 as the domain of f, the computation proceeds analogically. The function $f\,|_{J_2}$ is decreasing, therefore one-to-one, the formula for f^{-1} differs only by a minus sign from taking the negative square root on the next-to-last line,

$$y = -\sqrt{\frac{2}{3 - x}} - 1$$

and $I_{f^{-1}} = (-\infty, -1) = J_2$.

160 – Transposing formulae

Exercise

Decide if the given functions:

a) $p(b) : \frac{p}{q} = \sqrt{\frac{a+2b}{a-2b}}$ b) $f(x) : y = \frac{4x-1}{x+3}$ c) $y(r) : y + x = \frac{r}{4+r}$

are one-to-one. Compute the transposed formula for its inverse function f^{-1}. Write down their domains and ranges.

Hints

Algorithm:

1. Find $D(f)$.
2. Decide on monotonicity.
3. Switch $x \leftrightarrow y$
4. Compute y.

161 – Transposing formulae

Exercise

For the given functions:

a) $f(x): y = \ln(1 - e^x)$

b) $f(x): y = 3 + 2 \cdot \arccos \frac{x}{2}$

c) $f(x): y = \ln(x - 1) - \ln(x + 1)$

d) $S(L): S = \sqrt{\frac{3d(L-d)}{8}}$

find their domains and decide if they are one-to-one. Compute the transposed formula for its inverse. Determine the domain and range of the inverse.

Hints

Algorithm:

1. Find $D(f)$.
2. Decide on monotonicity.
3. Switch $x \leftrightarrow y$
4. Compute y.

162 – Transposing formulae

Exercise

For the given functions:

a) $M(R) : M = \pi(R^4 - r^4)$

b) $f(x) : y = \dfrac{1}{\sin(x)}$

find the biggest interval, where the function is one-to-one. On this interval compute the transposed formula for its inverse function f^{-1}. Determine the domain and range of the inverse.

Hints

Algorithm:

1. Find $D(f)$.
2. Decide on monotonicity.
3. Switch $x \leftrightarrow y$
4. Compute y.

163 – L'Hôpital rule

When two functions approach zero, their ratio might do anything. That is why we call $0/0$ an **indeterminate expression**. The results depends on the particular form of the expression hidden behind the zeros in the numerator and the denominator. We might have

$$\lim_{h\to 0}\frac{h^2}{h}=0 \quad\text{or}\quad \lim_{h\to 0}\frac{h}{h}=1 \quad\text{or}\quad \lim_{h\to 0}\frac{7h}{h}=7 \quad\text{or}\quad \lim_{h\to 0}\frac{\sqrt{h}}{h}=\infty.$$

What only matters is *whether the numerator goes to zero more quickly than denominator*.
There are eight typical indeterminate expressions:

$$\frac{0}{0},\quad \frac{\pm\infty}{\pm\infty},\quad 0\cdot\infty,\quad \infty-\infty,\quad 0^0,\quad 0^\infty,\quad \infty^0,\quad 1^\infty.$$

The efficient and powerful method to compute the limits of the indeterminate expressions is named afted Guillaume François Antoine, Marquis de l'Hôpital (1661–1704), who published it first in 1696. However the idea belongs probably to Jacob Bernoulli (1667–1748).

Theorem (L'Hôpital)

Suppose that

$$\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x)=0$$

or

$$\lim_{x\to x_0} f(x)=\pm\infty \quad\text{and}\quad \lim_{x\to x_0} g(x)=\pm\infty.$$

Then it holds

$$\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=a \qquad\Rightarrow\qquad \lim_{x\to x_0}\frac{f(x)}{g(x)}=a$$

Example

Compute the limits

a) $\displaystyle\lim_{h\to 0}\frac{\cos h-1}{h}$ b) $\displaystyle\lim_{h\to 0}\frac{\sin h}{h}$

used for the deduction of the rules for the derivative of sine and cosine functions.

Both expressions are of the form $0/0$, so we can use the l'Hôpital rule.
a) Formally, we should proceed carefully and start by computing the limit of the ratio of the derivatives:

$$\lim_{h\to 0}\frac{(\cos h-1)'}{(h)'}=\lim_{h\to 0}\frac{-\sin h}{1}=\sin(0)=0.$$

If this limit exists, then the original limit also exists and they are the same:

$$\lim_{h\to 0}\frac{(\cos h-1)'}{(h)'}=0 \qquad\Rightarrow\qquad \lim_{h\to 0}\frac{\cos h-1}{h}=0.$$

b) In the most cases the limit exists, even if perhaps after multiple derivatives. Therefore, we simplify the formal procedure:

$$\lim_{h\to 0}\frac{\sin h}{h}\overset{\text{l'H}}{=}\lim_{h\to 0}\frac{\cos h}{1}=\cos(0)=1.$$

164 – L'Hôpital rule

Example

Compute the limit
$$\lim_{x\to\infty} \frac{x+\sin x}{x}.$$

The expression is of the form ∞/∞, so we can apply l'Hôpital rule:

$$\lim_{x\to\infty} \frac{x+\sin x}{x} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{1+\cos x}{1} = 1+\cos(\infty).$$

This limit does not exist, as the cosine function oscillates for $x \to \infty$. However, from this fact we cannot deduce that the original limit does not exist! Just, in this rare case, l'Hôpital rule did not give us any answer and we should proceed with a different method:

$$\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} \frac{x}{x} + \frac{\sin x}{x} = 1 + \lim_{x\to\infty} \frac{\sin x}{x} = 1,$$

because the last limit is zero, as the limit of the expression of the form n/∞, with $n \in [-1,1]$.

Example

Show that the growth of the exponential function $y = \mathrm{e}^x$ is quicker than the growth of the power function $y = x^n$ for any $n \in \mathbb{N}$.

We compute limit for $x \to \infty$ of their ratio. We start with simple linear function x^1:

$$\lim_{x\to\infty} \frac{\mathrm{e}^x}{ax+b} = \text{“}\left(\tfrac{\infty}{\infty}\right)\text{”} = \lim_{x\to\infty} \frac{(\mathrm{e}^x)'}{(ax+b)'} = \lim_{x\to\infty} \frac{\mathrm{e}^x}{a} = \frac{1}{a} \lim_{x\to\infty} \mathrm{e}^x = \infty.$$

Hence the exponential function in the numerator grows quicker than the linear function in the denominator.
For a quadratic function we proceed in analogy with the previous calculation:

$$\begin{aligned}\lim_{x\to\infty} \frac{\mathrm{e}^x}{ax^2+bx+c} &= \text{“}\left(\tfrac{\infty}{\infty}\right)\text{”} = \lim_{x\to\infty} \frac{(\mathrm{e}^x)'}{(ax^2+bx+c)'} = \lim_{x\to\infty} \frac{\mathrm{e}^x}{2ax+b} \\ &= \text{“}\left(\tfrac{\infty}{\infty}\right)\text{”} = \lim_{x\to\infty} \frac{\mathrm{e}^x}{2a} = \frac{1}{2a} \lim_{x\to\infty} \mathrm{e}^x = \infty.\end{aligned}$$

For any power function $y = x^n$ with arbitrary $n \in \mathbb{N}$ we get the same result, however only after n derivatives.

165 – L'Hôpital rule

Exercise

Solve:

a) $\lim\limits_{x\to 0} \dfrac{\sin(4x)}{x}$

b) $\lim\limits_{x\to 0} \dfrac{\tan^2(2x)}{2x^2}$

Hints

Limits of the type "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$"

If $\lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)} = a$ then $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = a$

Alternatively, utilize the formula

$\lim\limits_{x\to x_0} \dfrac{\sin(x)}{x} = 1$

166 – L'Hôpital rule

Exercise

Solve:

a) $\lim\limits_{x \to \infty} \dfrac{1 - e^{2x}}{x^4}$

b) $\lim\limits_{x \to \infty} \dfrac{\ln^2 x}{x^2 + 1}$

Hints

Limits of the type "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$"

If $\lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)} = a$ then $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = a$

167 – L'Hôpital rule

Exercise

Solve:

a) $\lim\limits_{x\to\infty} \dfrac{4x^2+6x+9}{x^4+2x^2+1}$

b) $\lim\limits_{x\to\infty} \dfrac{\sqrt[3]{8x^3-1}}{(x-2)^2}$

Hints

Limits of the type "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$"

If $\lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)} = a$ then $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = a$

Alternatively, you can factor out the highest power and discuss

$$\lim_{x\to\infty} \frac{ax^m}{bx^n} = \begin{cases} \infty & \text{for } m > n \\ \frac{a}{b} & \text{for } m = n \\ 0 & \text{for } m < n \end{cases}$$

168 – L'Hôpital rule

Exercise

Solve:

a) $\lim\limits_{x \to \infty} \dfrac{x^3 - 7x^2 - 1}{6x^2 + 9x + 15}$

b) $\lim\limits_{x \to \infty} \dfrac{\sqrt{9x^4 + 8x^2 + 1}}{3x - 1}$

Hints

Limits of the type "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$"

If $\lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)} = a$ then $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = a$

Alternatively, you can factor out the highest power and discuss

$$\lim_{x \to \infty} \frac{ax^m}{bx^n} = \begin{cases} \infty & \text{for } m > n \\ \frac{a}{b} & \text{for } m = n \\ 0 & \text{for } m < n \end{cases}$$

169 – L'Hôpital rule

Exercise

Solve:

a) $\lim\limits_{x\to\infty} \dfrac{4x^3-1}{7x^3+6x^2+5x+4}$

b) $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^4-4x^2}}{3x^2+2x+1}$

Hints

Limits of the type "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$"

If $\lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)} = a$ then $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = a$

Alternatively, you can factor out the highest power and discuss

$$\lim_{x\to\infty} \frac{ax^m}{bx^n} = \begin{cases} \infty & \text{for } m > n \\ \frac{a}{b} & \text{for } m = n \\ 0 & \text{for } m < n \end{cases}$$

170 – One-sided limits

Sometimes we can approach the point, where we need to compute the limit, just from one side, e.g. due to the restrictions in the domain of the function. That is what we call **one-sided limit**. We usually speak about **limit from the right**, denote it by the small plus in the superscript $x \to x_0^+$, and **limit from the left**, denoted analogically by $x \to x_0^-$. We also often speak about left neighborhood $N_-(x_0) = (x_0 - \delta, x_0)$ and right neighborhood $N_+(x_0) = (x_0, x_0 + \delta)$ of theo point x_0.
Strictly speaking, the limits for $x \to \infty$ and $x \to -\infty$, can be also regarded as one sided limits.

Theorem

The function f has the limit L for $x \to x_0$ if and only if

$$\lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^+} f(x).$$

The one sided limits are used mostly for the computing the expressions $a/0$:

We say that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}, \qquad \text{with } \lim_{x \to x_0} f(x) = a \text{ and } \lim_{x \to x_0} g(x) = 0$$

is an expression of the form $a/0$. For its value it holds:

$$\frac{a}{0} = \infty \qquad \begin{cases} a > 0 & \text{and } g(x) \text{ positive} \\ a < 0 & \text{and } g(x) \text{ negative} \end{cases}$$

$$\frac{a}{0} = -\infty \qquad \begin{cases} a > 0 & \text{and } g(x) \text{ negative} \\ a < 0 & \text{and } g(x) \text{ positive} \end{cases}$$

Example

Compute the limit

$$\lim_{x \to 0^+} \frac{\sqrt{x}}{x^2}.$$

As $D = (0, \infty)$, we can only compute the limit at $x = 0$ from the right only. The expression is of the form $0/0$ and we can use l'Hôpital rule, as usual:

$$\lim_{x \to 0^+} \frac{\sqrt{x}}{x^2} \overset{\text{l'H}}{=} \lim_{x \to 0^+} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{2x} = \lim_{x \to 0^+} \frac{1}{4x\sqrt{x}} = \text{“}\frac{1}{+0}\text{”} = \infty.$$

Example

Compute the limit

$$\lim_{x \to 1} \frac{1}{x^2 - 1}.$$

The limit is of the form $1/0$ and the denominator changes sign at $x = 1$. Therefore, we compute the one-sided limits:

$$\lim_{x \to 1^-} \frac{1}{x^2 - 1} = \frac{1}{0} = -\infty,$$

as the denominator is negative on a left neighborhood, e.g. $N_-(1) = (0, 1)$,

$$\lim_{x \to 1^+} \frac{1}{x^2 - 1} = \frac{1}{0} = +\infty,$$

as the denominator is positive on a right neighborhood, e.g. $N_+(1) = (1, 8)$.
As the one sided limits have different values, the standard limit, $\lim_{x \to 1} \frac{1}{x^2-1}$ does not exist.

171 – L'Hôpital rule

The indeterminate expressions not covered by l'Hôpital rule may be rearranged to the form $\frac{\pm\infty}{\pm\infty}$ or $\frac{0}{0}$.
We start with the expressions of the form $0 \cdot \infty$. For them it holds:

$$0 \cdot \infty = 0 \cdot \frac{1}{\frac{1}{\infty}} = \frac{0}{0} \qquad \text{or} \qquad 0 \cdot \infty = \frac{1}{\frac{1}{0}} \cdot \infty = \frac{\infty}{\infty}.$$

Which one should we use? Generally speaking the one, which provides nicer expression for the derivative.

Example

Compute $\lim\limits_{x \to 0^+} x \cdot \ln x$.

The expressions of the form $0 \cdot (-\infty) = (-1) \cdot 0 \cdot \infty$. We show the both ways of tranformation:

$$\begin{aligned} \lim_{x \to 0^+} x \cdot \ln x &= \lim_{x \to 0^+} \frac{1}{\frac{1}{x}} \cdot \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \\ &\overset{\text{l'H}}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{(-1)x^{-2}} = \lim_{x \to 0^+} -x = 0 \end{aligned}$$

The other possibility

$$\begin{aligned} \lim_{x \to 0^+} x \cdot \ln x &= \lim_{x \to 0^+} x \cdot \frac{1}{\frac{1}{\ln x}} = \lim_{x \to 0^+} \frac{x}{\frac{1}{\ln x}} \\ &\overset{\text{l'H}}{=} \lim_{x \to 0^+} \frac{1}{(-1)(\ln x)^{-2}\frac{1}{x}} = \lim_{x \to 0^+} (-x) \cdot \ln^2 x \end{aligned}$$

leads to nowhere.

We turn to the expression $\infty - \infty$. The infinities most often arise when we divide by the variable x anf let x go to zero. The expression $0/0$ can be handled by l'Hôpital rule. For the expressions with non-zero numerator, i.e. $a/0$ we use the procedure prom the previus page, $a/0 = \pm\infty$.
That is why the expression $\infty - \infty$ may be usually expanded with a common denominator, which transforms it into either $0/0$ or $a/0$.

Example

Compute $\lim\limits_{x \to 1} \left(\frac{1}{\ln x} - \frac{2}{x-1} \right)$.

First we rewrite the bracket using the common denominator $(x-1)\ln x$:

$$\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \to 1} \left(\frac{x-1-\ln x}{(x-1)\ln x} \right),$$

which is of the form $0/0$, and next we can use L'Hôpital rule

$$\begin{aligned} \lim_{x \to 1} \left(\frac{x-1-\ln x}{(x-1)\ln x} \right) &\overset{\text{l'H}}{=} \lim_{x \to 1} \left(\frac{1-\frac{1}{x}}{\ln x + (x-1)\frac{1}{x}} \right) = \left(\frac{0}{0} \right) \\ &\overset{\text{l'H}}{=} \lim_{x \to 1} \left(\frac{-\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} \right) = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2} \end{aligned}$$

172 – L'Hôpital rule

Exercise

Compute the following limits:

a) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+2x}-2x\right)$

b) $\lim\limits_{x\to 0} x\cdot\cot(x)$

c) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x-1}-x\right)$

d) $\lim\limits_{x\to-\infty} x\cdot \mathrm{e}^x$

Hints

Limits of the type "$\infty-\infty$" and "$0\cdot\infty$" may be rearranged to the type "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$".

If $\lim\limits_{x\to x_0}\dfrac{f'(x)}{g'(x)}=a$ then $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=a$.

173 – Asymptotes

The asymptotes are special tangent lines that meet the graph at infinity. This is why we need to use limits to decide about asymptotes. We have three types of asymptotes:

Definition

Vertical asymptote $a_V : x = x_0$:

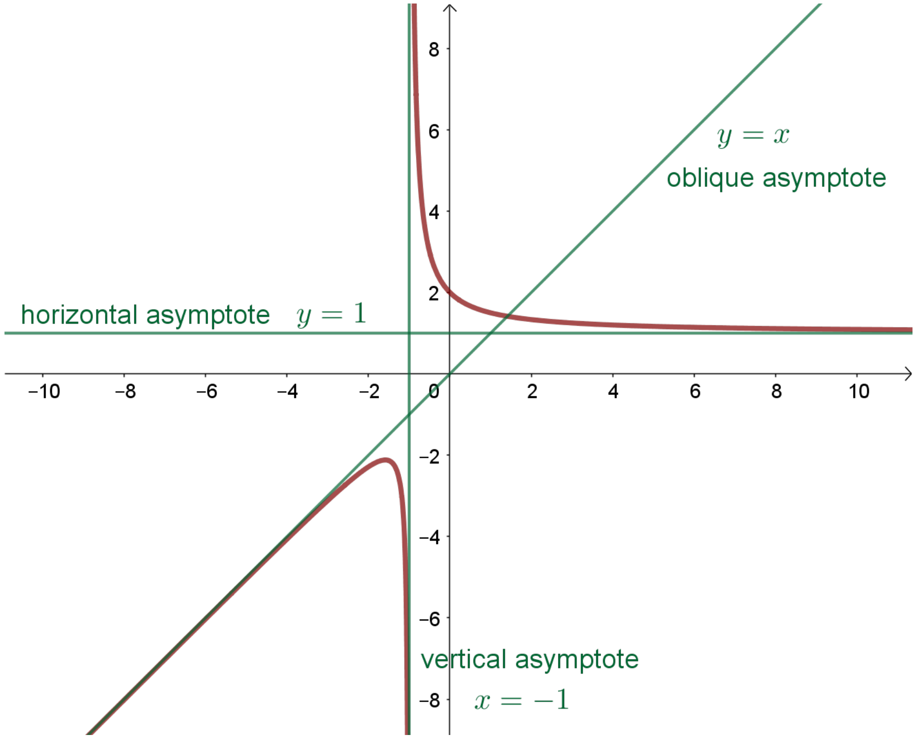
$$\lim_{x \to x_0^\pm} f(x) = \pm\infty$$

Horizontal asymptote $A_H : y = a$:

$$\lim_{x \to \infty} f(x) = a$$

Oblique asymptote $a_O : y = k \cdot x + q$:

$$\begin{aligned} k &= \lim_{x \to \infty} \frac{f(x)}{x} \\ q &= \lim_{x \to \infty} f(x) - k \cdot x \end{aligned}$$



The three types of asymptotes.

Remark

a) We note that when f is elementary continuous function, the only possibility of a vertical asymptote arises at the border points of the domain intervals $D = (a, b)$. Therefore we require only the existence of one one-sided limit for the existence of the vertical asymptote.

b) In some cases, there might exist a different oblique asymptote for $x \to -\infty$.

174 – Asymptotes

Example

Write down the equations of all asymptotes to the graph of the function

$$f : y = \frac{x^2 - 1}{x}.$$

We first determine the domain of f. Due to the x in the denominator, we have $D = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$.

Therefore, the existence of the only possible vertical asymptote $a_V : x = 0$ will be determined by the limit

$$\lim_{x \to 0} \frac{x^2 - 1}{x} = \text{“}\left(\tfrac{-1}{0}\right)\text{”} \pm \infty$$

As the one-sided limit from the left is ∞ and from the right $-\infty$, the asymptote indeed exists.

Next, we decide if there is horizontal or oblique asymptote. As

$$\lim_{x \to \infty} \frac{x^2 - 1}{x} \overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{2x}{1} = \infty$$

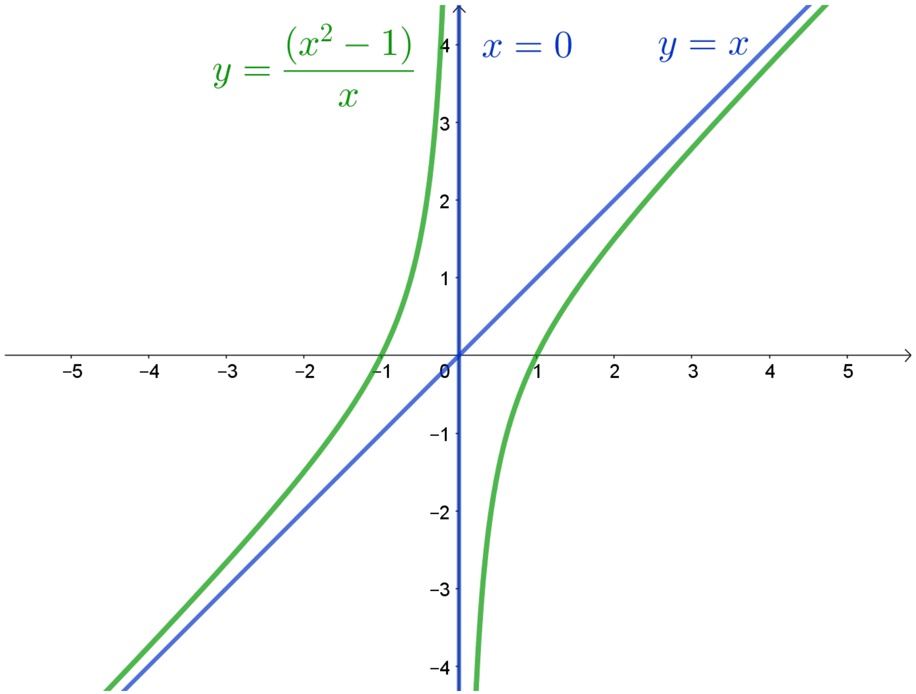
there is no horizontal asymptote.

Finally, we compute the formula for the oblique asymptote:

$$\begin{aligned} k &= \lim_{x \to \infty} \frac{\frac{x^2-1}{x}}{x} = \lim_{x \to \infty} \frac{x^2 - 1}{x^2} \overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{2x}{2x} = 1 \\ q &= \lim_{x \to \infty} \frac{x^2 - 1}{x} - 1 \cdot x = \text{“}(\infty - \infty)\text{”} = \lim_{x \to \infty} \frac{x^2 - 1 - x^2}{x} = \lim_{x \to \infty} \frac{-1}{x} = 0 \end{aligned}$$

The same holds also for $x \to -\infty$ and we have unique oblique asymptote $y = x$.

The situation is nicely visible form the graph:



The function $f : y = \dfrac{x^2 - 1}{x}$ and its asymptotes.

175 – Asymptotes

Example

Write down the equations of all asymptotes to the graph of the function

$$f : y = \arctan\left(\frac{1}{x}\right).$$

We first determine the domain of f. Due to the $1/x$ in the argument we have $D = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$.

Therefore, the existence of the only possible vertical asymptote $a_V : x = 0$ will be determined by the limit

$$\lim_{x\to 0} \arctan\left(\frac{1}{x}\right) = \text{“}\arctan\left(\tfrac{1}{0}\right) = \arctan(\pm\infty)\text{”}$$

We treat the cases separately. First, we take $x > 0$ and substitute $y = 1/x$. We get

$$\lim_{x\to 0^+} \arctan\left(\frac{1}{x}\right) = \lim_{y\to\infty} \arctan(y) = \frac{\pi}{2},$$

as the function arctan is increasing and $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. Similarly,

$$\lim_{x\to 0^-} \arctan\left(\frac{1}{x}\right) = \lim_{y\to-\infty} \arctan(y) = -\frac{\pi}{2}.$$

Hence, the one-sided limits are different and finite. Therefore, the vertical asymptote does not exist.

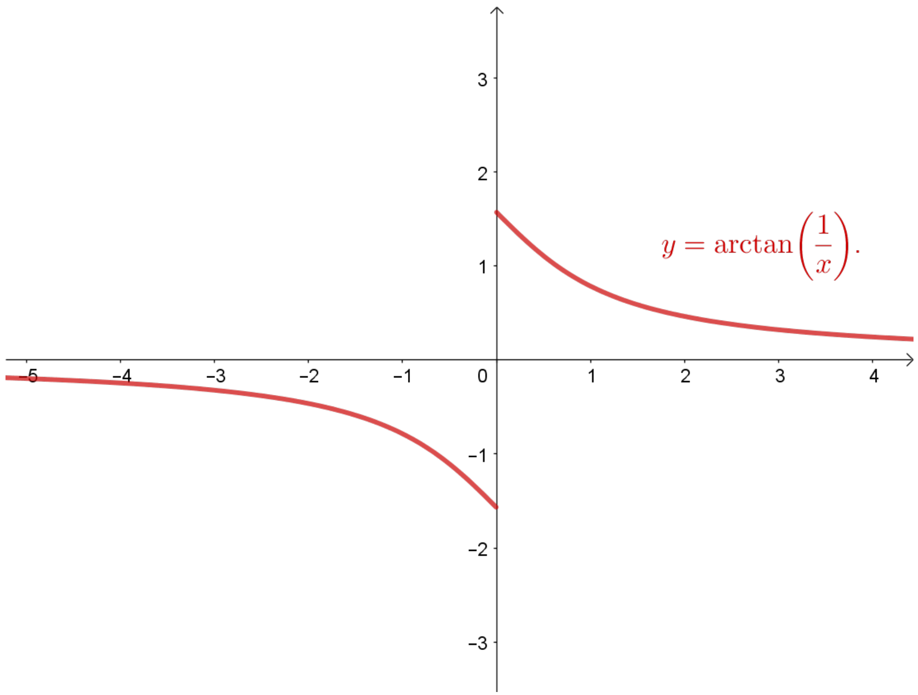
Next, we decide if there is horizontal or oblique asymptote. As

$$\lim_{x\to\infty} \arctan\left(\frac{1}{x}\right) = \arctan(0) = 0,$$

we have the horizontal asymptote $y = 0$ and there is no oblique asymptote.

Exercise

Draw the asymptotes into the graph of the function $f : y = \arctan\left(\frac{1}{x}\right)$.



Graph of the function $f : y = \arctan\left(\frac{1}{x}\right)$.

176 – Asymptotes

Exercise

Write down the equations of all asymptotes to the graph of the functions:

a) $y = \dfrac{x-4}{2x+6}$

b) $y = \dfrac{1}{x^2+x-2}$

c) $y = \dfrac{x+1}{x^2-4}$

d) $y = \dfrac{x^2-2x+2}{3x-4}$

Hints

Vertical asymptote $x = x_0$:
$\lim\limits_{x \to x_0^\pm} f(x) = \pm\infty$

Horizontal asymptote $y = a$:
$\lim\limits_{x \to \infty} f(x) = a$

Oblique asymptote $y = k \cdot x + q$:

$$k = \lim_{x \to \infty} \frac{f(x)}{x}$$

$$q = \lim_{x \to \infty} f(x) - k \cdot x$$

177 – Asymptotes

Exercise

Write down the equations of all asymptotes to the graph of the functions:

a) $y = \dfrac{x^2}{1-x}$

b) $y = 3 - 2x + \dfrac{1}{x^2}$

c) $y = \dfrac{1-x^2}{x^2+3x+4}$

d) $y = \dfrac{x^3+3x^2+1}{x^2+2}$

Hints

Vertical asymptote $x = x_0$:
$\lim\limits_{x \to x_0^\pm} f(x) = \pm\infty$

Horizontal asymptote $y = a$:
$\lim\limits_{x \to \infty} f(x) = a$

Oblique asymptote $y = k \cdot x + q$:

$$k = \lim_{x \to \infty} \frac{f(x)}{x}$$

$$q = \lim_{x \to \infty} f(x) - k \cdot x$$

178 – Asymptotes

Exercise

Write down the equations of all asymptotes to the graph of the functions:

a) $y = x \cdot \mathrm{e}^{-2x}$

b) $y = x^2 \cdot \mathrm{e}^{-x}$

c) $y = \ln \dfrac{x+1}{x-1}$

d) $y = \dfrac{\sin x}{x}$

Hints

Vertical asymptote $x = x_0$:
$\lim\limits_{x \to x_0^{\pm}} f(x) = \pm\infty$

Horizontal asymptote $y = a$:
$\lim\limits_{x \to \infty} f(x) = a$

Oblique asymptote $y = k \cdot x + q$:

$$k = \lim_{x \to \infty} \frac{f(x)}{x}$$

$$q = \lim_{x \to \infty} f(x) - k \cdot x$$

179 – Asymptotes

Exercise

Write down the equations of all asymptotes to the graph of the functions:

a) $y = \dfrac{\cos x}{x}$

b) $y = \dfrac{x}{2} - \cos x$

c) $y = x + \arctan \dfrac{x}{2}$

d) $y = \arctan \dfrac{x+1}{x}$

Hints

Vertical asymptote $x = x_0$:
$\lim\limits_{x \to x_0^{\pm}} f(x) = \pm\infty$

Horizontal asymptote $y = a$:
$\lim\limits_{x \to \infty} f(x) = a$

Oblique asymptote $y = k \cdot x + q$:

$$k = \lim_{x \to \infty} \frac{f(x)}{x}$$

$$q = \lim_{x \to \infty} f(x) - k \cdot x$$

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